A New Probabilistic Interpretation of Keller-Segel Model for Chemotaxis. Application to $d = 1$

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Outline

Chemotaxis

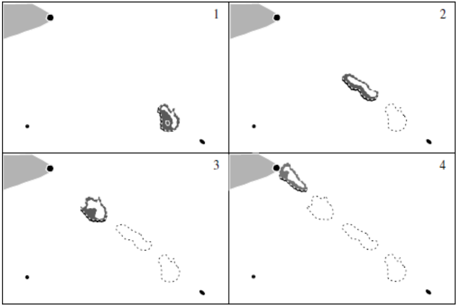

Figure: Motion of amoeba in reaction to chemo-attractant.

- Collective movement of a population (bacteria) in response to a chemical stimulus (food).
- Important role in many **biological processes**.
- Successfully modeled by the **Keller-Segel model** (K-S).

Keller-Segel model

- Coupled non linear system on **population density** - $\rho(t,x)$ and **chemo-attractant concentration** - $c(t,x)$:

$$\left\{\begin{array}{l} \partial_t \rho(t,x) = \nabla\cdot(\nabla\rho - \chi\rho\nabla c), \quad t>0,\ x\in\mathbb{R}^d, \\ \alpha\partial_t c(t,x) = \triangle c - \lambda c + \rho, \quad t>0,\ x\in\mathbb{R}^d. \\ \rho(0,x) = \rho_0(x), c(0,x) = c_0(x), \end{array}\right. \tag{1}$$

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- $\alpha = 0$ - parabolic-elliptic, $\alpha = 1$ - parabolic-parabolic model.
- $\chi > 0$ - **chemo-attractant sensitivity**.
- **Mass conservation**: $\int_\Omega \rho(t,x)dx = \int_\Omega \rho_0(x)dx := M$.

K-S model - well-posedness

- The subject of a huge amount of PDE literature over the last 30 years.
    - We refer to Perthame (2004) for a survey paper.

- Solutions may **blow-up in finite time:**
  for a $T < \infty$: $\lim_{t \to T} \sup(\|\rho_t\|_\infty + \|c_t\|_\infty) = +\infty$

- Global existence or the blow-up in finite time: space dimension dependent phenomenon.

Parabolic-parabolic case - well-posedness

- All authors consider: **positive solutions of weak type.**

- Case $d = 1$: **global existence** on bounded intervals I for both types of K-S model (Osaki-Yagi (2001) and Hillen-Potapov(2004)).

- Case $d = 2$: the **"threshold"** phenomenon:
  - $\boldsymbol{M \cdot \chi < 8\pi}$ - global existence.
  - Exists a solution with a blow up for $\boldsymbol{M \cdot \chi > 8\pi}$.
  - Global existence even with $\boldsymbol{M \cdot \chi > 8\pi}$ under additional conditions on (ρ_0, c_0) and size of α.

(Mizogouchi(2013) , Corrias et al.(2014) and the references therein)

Parabolic-parabolic case - well-posedness $d > 2$

- Blowing-up solutions exist for any positive mass.
- Global existence condition involves small $\|\rho_0\|_{L^{d/2}}$.
- **Not clear** if the blow-up in finite time follows from large $\|\rho_0\|_{L^{d/2}}$.

(Corrias-Perthame(2006) and the references therein.)

Outline

Probabilistic side - literature

- Parabolic-elliptic ($\alpha = \mathbf{0}$) model in $d = 2$:

  Haskovec-Schmeiser (2011) and Fournier-Jourdain (2016).

- Fournier-Jourdain (2016) give rise to the NLSDE:

$$dX_t = \sqrt{2}dW_t + \chi(K \star \rho_t)(X_t)dt,$$

  where $K(x) = -\frac{x}{2\pi|x|^2}$.
  For $\chi < 2\pi$, associated particle system is well defined, μ_N is tight and any weak limit is the law of X_t.

Our objectives

- Propose a **new probabilistic interpretation** for the fully parabolic model.
- Any space dimension.
- Theoretical and numerical viewpoint

Informal explanation of our interpretation I

$$\begin{cases} \partial_t \rho(t,x) = \nabla \cdot (\nabla \rho - \chi \rho \nabla c), \quad t > 0,\ x \in \mathbb{R}^d, \\ \partial_t c(t,x) = \triangle c - \lambda c + \rho, \quad t > 0,\ x \in \mathbb{R}^d. \\ \rho(0,x) = \rho_0(x), c(0,x) = c_0(x), \end{cases}$$

- We will aim for the integral solutions that verify

$$\begin{cases} \rho_t = g_t * \rho_0 + \chi \int_0^t \nabla g_{t-s} * (\rho_s \nabla c_s) ds \\ c_t = e^{-\lambda t} g_t * c_0 + \int_0^t e^{-\lambda s} \rho_{t-s} * g_s ds. \end{cases}$$

  g - density of $\sqrt{2} W_t$.

- Compute ∇c_t:

$$\nabla c_t = e^{-\lambda t} g_t * \nabla c_0 + \int_0^t e^{-\lambda(t-s)} \rho_s * \nabla g_{t-s} ds.$$

- Plug ∇c_s in the equation for ρ_t.

Our interpretation

- ρ_t is the **density** of stochastic process:

$$dX_t = \sqrt{2}dW_t + \quad \chi e^{-\lambda t} \int \nabla_x c_0(x + \sqrt{2}y)|_{x=X_t} g(t,y)dy \quad dt$$
$$+ \quad \chi \int_0^t \int e^{-\lambda(t-s)} \rho_s(y) \nabla_x g(t-s, \frac{y-x}{\sqrt{2}})|_{x=X_t} dyds \quad dt.$$

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  - Singular kernel, convolution in **time and space**.

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  - Singular kernel, convolution in **time and space**.

- **Concentration:**

$$c(t,x) = e^{-\lambda t}\mathbb{E}(c_0(x+\sqrt{2}W_t)) + \mathbb{E}\int_0^t e^{-\lambda s}\rho_{t-s}(x+\sqrt{2}W_s)ds.$$

Come back to the K-S model:

1. Construct the family $(\rho_t)_{t \leq T}$.
2. Construct the family $(c_t)_{t \leq T}$.
3. Prove the pair (ρ_t, c_t) **solves K-S**.

- Precise the **notion of solution** to K-S.

One-dimensional case

1. NLSDE becomes:

$$\left\{ \begin{array}{l} dX_t = \sqrt{2} dW_t + b(t, X_t) dt \\ +\chi C \int_0^t \int e^{-\lambda(t-s)} \frac{y-X_t}{(t-s)^{3/2}} e^{-\frac{(y-X_t)^2}{4(t-s)}} \rho_s(y) dy ds dt \\ X_0 \sim \rho_0, X_s \sim \rho_s(y) dy. \end{array} \right. \quad (2)$$

- ***b* linear part**.
- well-defined?

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3. (ρ_t, c_t) unique solution to the KS system?

Notion of solution

Definition 1

Let $\chi > 0$ *and* $T > 0$ *be given. The pair* (ρ, c) *is said to be a solution to K-S model if*

$$\rho_t \in L^1(\mathbb{R}), \quad \sqrt{t}\|\rho_t\|_{L^\infty(\mathbb{R})} \leq C, \quad c_t \in C_b^1(\mathbb{R}) \text{ for } t \in (0, T),$$

and $\forall t \in (0, T]$

$$\begin{cases} \rho_t = g(t, \cdot) * \rho_0 + \chi \int_0^t \frac{\partial}{\partial y} g(t-s, \cdot) * (\rho_s \frac{\partial}{\partial x} c(s, \cdot\ ))ds \\ c_t = e^{-\lambda t} g(t, \cdot) * c_0 + \int_0^t e^{-\lambda s} \rho_{t-s} * g(s, \cdot) ds. \end{cases}$$

- g - density of $\sqrt{2}W_t$.

Main results, d=1

Theorem 1 (Talay, T.)

Assume that $\rho_0 \in L^1(\mathbb{R})$ *and* $c_0 \in C_b^1(\mathbb{R})$. *Then, for any* $T > 0$ *and* $\chi > 0$, *Equation* (2) *admits a unique weak solution up to* T *in the class of measures on* $C[0, T]$ *whose one dimensional marginals are densities* $(p_t)_{t \leq T}$ *which satisfy for any* $t \in (0, T)$

$$\|p_t\|_\infty \leq \frac{C_T}{\sqrt{t}}. \tag{3}$$

- Proof of Theorem 1 can be rewritten with the **additional assumption**:
  - $\rho_0 \in L^\infty(\mathbb{R})$. Then, Theorem 1 is valid with a change of space for ρ: $\rho \in L^\infty([0, T]; L^1 \cap L^\infty(\mathbb{R}))$.
  - $\rho_0 \in L^2(\mathbb{R})$. Then, Theorem 1 is valid with a change of space for ρ: $\rho_t \in L^1(\mathbb{R}))$ and $t^{1/4}\|\rho_t\|_{L^\infty(\mathbb{R})} \leq C$.

Main results, d=1

Theorem 2 (Talay, T.)

Let $T > 0$ and $\chi > 0$. Assume that $\rho_0 \in L^1(\mathbb{R})$ and $c_0 \in C_b^1(\mathbb{R})$. Let $(\rho_t)_{t \leq T}$ be the family constructed in Theorem 1. Define $(c_t)_{t \leq T}$ as:

$$c(t,x) = e^{-\lambda t}\mathbb{E}(c_0(x + \sqrt{2}W_t)) + \mathbb{E}\int_0^t e^{-\lambda s}\rho_{t-s}(x + \sqrt{2}W_s)ds.$$

Then, the pair $(\rho_t, c_t)_{t \leq T}$ is a unique solution to the KS system in the sense of Definition 1.

Main results, d=1

Theorem 2 (Talay, T.)

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PDE results:

- Osaki-Yagi (2001): $\rho_0 \in L^2(I) \cap L^1(I), c_0 \in H^1(I)$ and $\inf_{x\in I} c_0 > 0$.

Main results, d=1

Theorem 2 (Talay, T.)

Let $T > 0$ and $\chi > 0$. Assume that $\rho_0 \in L^1(\mathbb{R})$ and $c_0 \in C_b^1(\mathbb{R})$. Let $(\rho_t)_{t \leq T}$ be the family constructed in Theorem 1. Define $(c_t)_{t \leq T}$ as:

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PDE results:

- Osaki-Yagi (2001): $\rho_0 \in L^2(I) \cap L^1(I), c_0 \in H^1(I)$ and $\inf_{x \in I} c_0 > 0$.
- Hillen-Potapov (2004): $\rho_0 \in L^\infty(I) \cap L^1(I), c_0 \in W_p^\sigma(I), (\sigma, p) \in A$.

Theorem 1: Iterative procedure

- We define the **sequence** X^k:
  step k=1:

$$\begin{cases} dX_t^1 = dW_t + \left\{ \int_0^t \int K(t-s, X_t^1 - y) p_0(y) dyds \right\} dt \\ X_0^1 \sim p_0. \end{cases}$$

  step k:

$$\begin{cases} dX_t^k = dW_t + \left\{ \int_0^t \int K(t-s, X_t^k - y) p_s^{k-1}(y) dyds \right\} dt \\ X_0^k \sim p_0. \end{cases}$$

- Tightness: bound all the drifts uniformly in k (density estimates) and $t < T_0$.
- Solve a non-linear martingale problem associated to the NLSDE.

Key argument in Theorem 1 - density estimates result

- We adapt the arguments of Qian-Zheng (2002).
- Let $X^{(b)}$:

$$\left\{ \begin{array}{l} dX_t^{(b)} = b(t, X_t^{(b)})dt + dW_t, \quad t \in [0, T], \\ X_0^{(b)} \sim p_0. \end{array} \right.$$

  Assume $\boldsymbol{\beta := \sup_{t\in[0,T]} \|b(t,\cdot)\|_\infty < \infty}$.
- We obtain the density estimate when $p_0 \in L^1(\mathbb{R})$ and $t > 0$:

$$\|p_t\|_\infty \leq \frac{C}{\sqrt{t}} + \beta.$$

Related Particle system

$$\begin{cases} dX_t^{i,N} = \sqrt{2}dW_t^i + \frac{\chi}{N}\sum_{j=1}^N \int_0^t K(t-s, X_t^{i,N} - X_s^{j,N})ds\mathbb{1}\{X_t^i \neq X_t^j\}dt \\ X_0^i \quad i.i.d. \sim p_0 \end{cases} \tag{4}$$

Non-Markovian, singularity in time and space

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Non-Markovian, singularity in time and space

Theorem 3 (Jabir, Talay, T.)

Let $T > 0$ and $\chi > 0$. Then, for a fixed $N \in \mathbb{N}$ there exists a weak solution to the system (4).

Theorem 3 - main arguments

- **Idea:** Procedure of Krylov-Röckner (2005) for constructing a solution to an SDE with singular coefficients.
- Let $B_t(x) := (B_t^1(x), \dots, B^N(x)), x \in \mathbb{R}^N$, where $B^i(x)$ is the drift of the equation for $X^{i,N}$.
- Let $\mathbb{Q}^N$ be the probability measure under which

$$\begin{cases} \bar{X}_t^{i,N} = \bar{X}_0^i + W_t^i, \quad t \leq T \\ \bar{X}_0^i \sim p_0 \text{ i.i.d.} \end{cases} \tag{5}$$

We prove that the Novikov condition is satisfied:

Proposition 1

Let $T > 0$, $\kappa > 0$ and $N \in \mathbb{N}$. Then,

$$\mathbb{E}_{\mathbb{Q}}\left(\exp\{\kappa \int_0^T |B_t|^2 dt\}\right) \leq C(T, \chi, N, \kappa)$$

Related Particle system

Theorem 4 (Jabir, Talay, T.)

The sequence of empirical measures $\mu^N := \frac{1}{N}\sum_{i=1}^N \delta_{X^{i,N}}$ *is tight.*

- Use the Kolmogorov criterion:
  $\mathbb{E}_{\mathbb{P}^N}[|X_t^{1,N} - X_s^{1,N}|^4] = \mathbb{E}_{\mathbb{Q}^N}[Z_T|\bar{W}_t^1 - \bar{W}_s^1|^4]$
  Problem: The estimate on the Girsanov term tends to infinity as $N \to \infty$.
  Solution: We perform a Girsanov transformation that involves just one particle.

Theorem 5 (Propagation of chaos result -Jabir, Talay, T.)

The sequence $\mathcal{L}\{\mu^N\}$ *converges weakly to* $\delta_{\mathbb{P}}$*, where* $\mathbb{P}$ *is the unique solution to the martingale problem related to the SDE related to the KS model.*

We adapt Bossy - Talay (1996) and use Girsanov transformations involving finite number of particles.

Outline

Current extensions and further objectives

- $d = 2$ - local existence and uniqueness $->$ not yet global. Particle system numerical simulations give hope!

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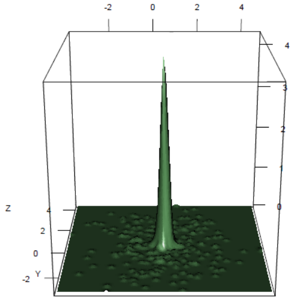
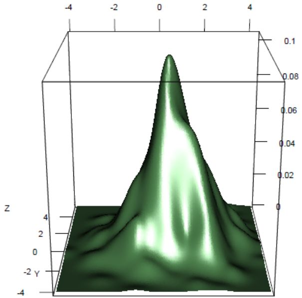
- $d = 2$ - local existence and uniqueness $->$ not yet global.
  Particle system numerical simulations give hope!
  Density time profiles for the simplified system:
  $t = 0$: uniform distribution,
  $t = 1$:



$\chi = 1$



$\chi = 0.2$

- $d \geq 3$.

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