Probabilistic interpretations of nonlinear parabolic systems

Belopolskaya Ya.

SPbSUACE, POMI RAN St.Petersburg

Snegiri December 2017

Supported by Grant RSF 17-11-01136

Aim : The aim of this talk is to consider systems of nonlinear Kolmogorov type equations arising in various applications in particular in physics, biology, population dynamics and banking and so on.
We are interested in stochastics underlying solution of the (**backward and/or forward**) Cauchy problem

$$f_s + F(x, f, \nabla f, \nabla^2 f) = 0, \quad f(T, x) = f_0(x),$$

or

$$g_t = F(x, g, \nabla g, \nabla^2 g), \quad g(0, x) = f_0(x)$$

where $f, g : [0, T] \times R^d \to R^{d_1}$, and
$F, G : R^d \times R^{d_1} \times M \times N \to R^{d_1}$, $M = R^d \otimes R^{d_1}$, $N = R^d \otimes R^d \otimes R^{d_1}$.

Nonlinear PDEs

Consider the Cauchy problem for a semilinear backward PDE

$$\mathbf{u_t} + \frac{1}{2}\mathbf{Tr A^u(x)}\nabla^2\mathbf{u[A^u]^*(x)} + \langle \mathbf{a^u(x)}, \nabla\rangle \mathbf{u} = \mathbf{0}$$

$$= \mathbf{u_t} + \mathcal{L}^{\mathbf{u}}\mathbf{u(t)} \qquad (1)$$

$\mathbf{u(T,x)} = \mathbf{u_0(x)}, \quad a^u(x) = a(x, u(x)).$

If $u \in C^{1,2}$ solves (1), then by Ito's formula $u(T, \xi^u(T)) - u(t,x) = \int_t^T [u_s + \mathcal{L}^u u](s, \xi^u(s))ds + \int_t^T \langle \nabla u(s, \xi^u(s)), A(\xi^u(s), u(s, \xi^u(s)))dw(s)\rangle$ and hence

$$\mathbf{u(t,x)} = \mathbf{E u_0(\xi^u_{t,x}(T))}$$

$$\mathbf{d}\xi^{\mathbf{u}}(\theta) = \mathbf{a^u}(\xi^{\mathbf{u}}(\theta))\mathbf{d}\theta + \mathbf{A^u}(\xi^{\mathbf{u}}(\theta))\mathbf{dw}(\theta), \quad \xi^{\mathbf{u}}(\mathbf{t}) = \mathbf{x}.$$

Consider the forward Cauchy problem

$$\mathbf{v_t} = \mathcal{L}^{\mathbf{v}}\mathbf{v(t)}, \quad \mathbf{v(0,x)} = \mathbf{u_0(x)}, \tag{2}$$

$\mathcal{L}^{\mathbf{v}}\mathbf{u} = \frac{1}{2}\mathbf{Tr A^v(x)}\nabla^2\mathbf{u}[\mathbf{A^v}]^*\mathbf{(x)} + \langle \mathbf{a^v(x)}, \nabla\rangle\mathbf{u}.$

There are two ways to construct a probabilistic representation to (2).

1) Assuming $u \in C^{1,2}$ solves (2) we choose $v(t,x) = u(T-t,x)$ and apply the above construction to $v(t,x)$.

But if we consider (2) as an equation for a measure density of a certain diffusion process then to get a generator of this process by definition we have to consider a dual equation derived from

$$-\int_{R^d} u_0(x)h(0,x)dx - \int_0^\infty \int_{R^d} u(t,x)\mathbf{h_t(t,x)}dxdt$$

$$= \int_0^\infty \int_{R^d} u[\frac{1}{2}\mathbf{Tr}\nabla^2[\mathbf{G^u(x)h}] - \langle \nabla, \mathbf{a^u(x)h}\rangle]dxdt$$

valid for any test function $h \in C_0^{1,2}([0,\infty) \times R^d)$. Here $G^u = A^u[A^u]^*$. Consider

$$\mathbf{h_t} + \frac{1}{2}\mathbf{Tr}\nabla^2[\mathbf{G^u(x)h}] - \langle \nabla, \mathbf{a^u(x)h}\rangle] = \mathbf{0}, \qquad (3)$$

Probabilistic representations of the Cauchy problem solutions for **scalar** **linear and nonlinear** parabolic equations

i) Classical sol. **MkKean (1966), Freidlin (1967,1980) Bel-Dalecky (80-90)**,

ii)Viscosity sol. **Pardoux, Peng, and others (90- till now)**

iii) Generalized sol. for **linear** PDEs

Kunita (1994), Bally, Matoussi (2001), Matoussi, Xu (2008) (forward and backward)

Bogachev (2000- 2010)

and for **nonlinear** PDEs

Bel-Woyczynski (2006-2007, 2013), Kolokoltsov (2005 –2010),

Nonilinear systems

Set $G^{ml}\kappa^l = \sum_{l=1}^{d_1} G^{ml}\kappa^l$ (**Summing over repeated indices**)

$$\frac{\partial u^m}{\partial s} + \frac{1}{2}G^{ml}_{ij}(x,u)\frac{\partial^2 u^l}{\partial x_i \partial x_j} + B^{lm}_i(x,u)\frac{\partial u^l}{\partial x_i} + g^m(x,u) = 0$$

$$u^m(T,x) = u^m_0(x). \qquad (4)$$

$$\frac{\partial u^m}{\partial t} = \frac{1}{2}G^{ml}_{ij}(x,u)\frac{\partial^2 u^l}{\partial x_i \partial x_j} + B^{lm}_i(x,u)\frac{\partial u^l}{\partial x_i} + g^m(x,u).$$

$$u^m(0,x) = u^m_0(x), \qquad (5)$$

I. Systems with diagonal principal part

1) $\mathcal{L}^u(x)v = \frac{1}{2}A^u_{jk}(x)\nabla^2_{ij}vA^u_{ki}(x) + a^u_i(x)\nabla_i v$

$$\mathbf{u^m_s} + \mathcal{L}^{\mathbf{u}}(\mathbf{x})\mathbf{u^m} + \mathbf{B^{lm}_i}(\mathbf{x},\mathbf{u})\frac{\partial \mathbf{u^l}}{\partial \mathbf{x_i}} + \mathbf{c^{lm}}(\mathbf{x},\mathbf{u})\mathbf{u^l} = 0.$$

$$\mathbf{u}(\mathbf{T},\mathbf{x}) = \mathbf{u_0}(\mathbf{x}), \qquad (6)$$

2) $B^{lm}_i(x,u) \equiv 0, \quad \mathcal{L}^u_m(x)v = \frac{1}{2}A^u_{jk}(x,m)\nabla^2_{ij}vA^u_{ki} + a^u_i(x,m)\nabla_i v$

$$\mathbf{u^m_s} + \mathcal{L}^{\mathbf{u}}_{\mathbf{m}}(\mathbf{x})\mathbf{u^m} + \sum_{\mathbf{l}=1}^{\mathbf{d_1}}\mathbf{c^{ml}}(\mathbf{x},\mathbf{u})\mathbf{u^l} = \mathbf{0},\ \mathbf{u}(\mathbf{T},\mathbf{x}) = \mathbf{u_0}(\mathbf{x}), \qquad (7)$$

$m = 1, \ldots, d_1$. No summation in m!

3) $B_i^{lm}(x, u) \neq 0$.

$$\mathbf{u_t^m} + \mathbf{B_i^{ml}}(\mathbf{x}, \mathbf{u}) \frac{\partial \mathbf{u^l}}{\partial \mathbf{x_i}} = \mathcal{L}_{\mathbf{m}}^{\mathbf{u}}(\mathbf{x}) \mathbf{u^m}, \ \mathbf{u}(\mathbf{0}, \mathbf{x}) = \mathbf{u_0}(\mathbf{x}). \tag{8}$$

MHD-Burgers

$$\frac{\partial \mathbf{v}}{\partial \mathbf{t}} + \langle \mathbf{v}, \nabla \rangle \mathbf{v} = \frac{\mathbf{1}}{\mathbf{2}} \nu^2 \mathbf{\Delta v} + (\nabla \times \mathbf{B}) \times \mathbf{B}, \quad \mathbf{v}(\mathbf{0}) = \mathbf{v_0}, \tag{9}$$

$$\frac{\partial \mathbf{B}}{\partial \mathbf{t}} = \frac{\mathbf{1}}{\sigma^2} \mathbf{\Delta B} + \nabla \times (\mathbf{v} \times \mathbf{B}), \quad \mathbf{B}(\mathbf{0}, \mathbf{x}) = \mathbf{B_0}(\mathbf{x}), \tag{10}$$

$B(t, x) \in R^3$ –magnetic field, $v(t, x) \in R^3$ - velocity, $\times$ – vector product, $t \in [0, \infty), x \in R^3$.

4) Systems with nondiagonal principal part

Population dynamics

$$\frac{\partial u^m}{\partial t} = \Delta(u^m[u^1 + u^2]) + c_u^m u^m, \qquad m = 1, 2, \quad (11)$$

$$u^1(0, x) = u_0^1(x), u^2(0, x) = u_0^2(x),$$

where

$$c_u^m = c_m - c_{m1} u^1 - c_{m2} u^2$$

and $c_m, c_{mk}, m, k = 1, 2$ are positive constants.

PDE results for systems of type :

(1) **Ladyzenskaya O, Solonnikov V., Uraltzeva N. (1967) Linear and quasilinear equations of parabolic type.**

(2) **D. H. Sattinger (1976)**

(3,4) **H. Amann (1989)**

Probabilistic representations of the Cauchy problem solutions for systems of type **(1)**

a) Classical solutions **Bel-Dalecky (80-90)**,

b) Viscosity solutions **Bel (2013-2015),**

c) Generalized solutions **Bel-Woyczynski (2011-2013)**,

(2) Classical solutions **M.Kac (1956) Griego, Hersh (1968), Freidlin (1980)**

Classical and viscosity sol. **Pardoux, Peng, Antonnely, Ma, Yong,..... (1992 -till now)**

Probabilistic representations of the Cauchy problem generalized solutions for systems of type **(3, 4). Bel (2015-2017)**

Mean field approach or limit of diffusive stochastic population dynamics

Fontbona J., Meleard S.(2015),
Galiano G., Selgas V., (2014)

1. i) **Semilinear case** $d_1 > 1$ The Cauchy problem

$$\mathbf{u}^{\mathbf{m}}_{\mathbf{s}} + \mathcal{L}^{\mathbf{u}}(\mathbf{x})\mathbf{u}^{\mathbf{m}} + \mathbf{B}^{\mathbf{ml}}(\mathbf{x},\mathbf{u})\nabla\mathbf{u}^{\mathbf{l}} + \mathbf{c}^{\mathbf{ml}}(\mathbf{x},\mathbf{u})\mathbf{u}^{\mathbf{l}} = \mathbf{0}, \tag{7}$$

$u(T,x) = u_0(x)$.

The correspondent stochastic problem has the form

$\xi(\mathbf{s}) = \mathbf{x} \in \mathbf{R}^{\mathbf{d}}$,

$$d\xi(\theta) = a(\xi(\theta), u(\theta,\xi(\theta)))d\theta + A(\xi(\theta), u(\theta,\xi(\theta)))dw(\theta), \tag{12}$$

$$d\eta(\theta) = c(\xi(\theta), u(\theta,\xi(\theta)))\eta(\theta)d\theta$$
$$+C(\xi(\theta), u(\theta,\xi(\theta)))(\eta(\theta), dw(\theta)), \quad \eta(s) = h \in R^{d_1}, \tag{13}$$

$$\langle h, u(s,x)\rangle = E[\langle \eta(T), u_0(\xi_{s,x}(T))\rangle], \tag{14}$$

where $B^{lm}_i = C^{lm}_j A_{ji}$.

Condition **C 1**

$a(x,u), A(x,u)$ satisfy Lipschitz condition in x and u, sublinear in x and have polynomial growth in u,

$$\|u_0\| = sup_x \|u(0,x)\| \le K_0, \quad \|u_x(0,x)\| \le K_0^1.$$

Condition **C 2** = Condition **C 1** +

$$2\langle c(x,v)h,h\rangle \le [\rho_0 + \rho\|v\|^2]\|h\|^2,$$

$$\|(C(x,v)h)\|^2 \le \rho[1+\|v\|^2]\|h\|^2,$$

$$2\langle [c(x,v)-c(x_1,v_1)]h,h\rangle + \|[C(x,v)-C(x_1,v_1)]h)\|^2$$

$$\le L_v\|x-x_1\|^2 + K(r)\|v-v_1\|^2]\|h\|^2,$$

where $C, \rho, L, L_1 L_{(v,v_1)} > 0$ and ρ_0- constants and $r = max(\|v\|, \|v_1\|)$.

Condition **C 3** = **C 1** + **C 2** for derivatives $a_x^{(k)}, A_x^{(k)}, c_x^{(k)}, C_x^{(k)}, k = 1, 2$.

Theorem

Let $u(s,x)$ be a classical solution to of the Cauchy problem (7) and **C 2** *holds. Then $u(s,x)$ admits a probabilistic representation of the form (14), where $\xi(t)$ and $\eta(t)$ satisfy (12) and (13).*

Theorem

Assume that there exists a solution to (12)-(14) and $u(s) \in C^2$. *Then* $u(s,x)$ *stands for the unique classical solution of (7).*

Theorem

Assume that **C 3** *holds for* $k = 1$ *and* $u_0 \in C^1$. *Then there exists an interval* $[\tau, T]$ *such that for* $s \in [\tau, T]$ *there exists a unique solution of (12)-(14).*

Theorem

Assume **C 3** *holds for* $k = 3$ *and* $u_0 \in C^2$. *Then there exists an interval* $[\tau_1, T] \subset [\tau, T]$ *such that for* $s \in [\tau_1, T]$ *the function* $u(s, x)$ *of the form (14) is the unique classical solution of (7).*

$$\frac{\partial \mathbf{v_m}}{\partial \mathbf{s}} + \mathcal{L}^{\mathbf{v}}_{\mathbf{m}} \mathbf{v_m} + [\mathbf{Q}^{\mathbf{v}} \mathbf{v}]_{\mathbf{m}} = \mathbf{0}, \quad \mathbf{v_m}(\mathbf{t}, \mathbf{x}) = \mathbf{v_{0m}}(\mathbf{x}), \tag{15}$$

$$\mathcal{L}^v_m v_m = a_m(x, v) \cdot \nabla v_m + \frac{1}{2} Tr A_m(x, v) \nabla^2 v_m A^*_m(x, v)$$

$$[Q^v v]_m = \sum_{j=1}^{d_1} q^v_{lm} v_m.$$

$$\mathbf{P}\{\gamma(\mathbf{t} + \mathbf{\Delta t}) = \mathbf{l} | \gamma(\mathbf{t}) = \mathbf{j}, (\xi(\theta), \gamma(\theta)), \theta \le \mathbf{t}\} = \mathbf{q^v_{jl}}(\xi(\mathbf{t})$$

$$\mathbf{d}\xi(\mathbf{t}) = \mathbf{a}^{\mathbf{v}}(\xi(\mathbf{t}), \gamma(\mathbf{t}))\mathbf{dt} + \mathbf{A}^{\mathbf{v}}(\xi(\mathbf{t}), \gamma(\mathbf{t}))\mathbf{dw}(\mathbf{t}), \xi(\mathbf{s}) = \mathbf{x} \quad (16)$$

$$\mathbf{d}\gamma(\theta) = \int_{\mathbf{R}} \mathbf{g}^{\mathbf{v}}(\xi(\theta), \gamma(\theta-), \mathbf{z})\mathbf{p}(\mathbf{d}\theta, \mathbf{dz}), \quad \gamma(\mathbf{s}) = \mathbf{l},$$

$$\mathbf{v}(\mathbf{s}, \mathbf{x}, \mathbf{m}) = \mathbf{E}[\mathbf{v_0}(\xi_{\mathbf{s},\mathbf{x}}(\mathbf{T}), \gamma_{\mathbf{s},\mathbf{m}}(\mathbf{T}))]. \quad (18)$$

$$g^v(x, l, z) = \begin{cases} m - l & if\ z \in \Delta^v_{lm}(x), \\ 0 & else \end{cases}$$

$\Delta^v_{lm}(x)$ are consecutive, left closed, right open intervals of the real line, each having length $q^v_{lm}(x)$.

Smooth solutions (Bel 16), viscosity solutions (PPR 1997)

Theorem

Assume that $u_0(m) \in C^2$ and coefficients $a^u_m, A^m(u), q^m(u) \in C^2$. Then the function $v(s,x,m)$ given by (18) is a unique classical solution of (15).

The common feature of systems (7) and (15) is that they are equivalent to scalar parabolic equations in new phase spaces

$$\frac{\partial \Phi}{\partial s} + \frac{1}{2} TrG(\kappa, u)\nabla^2 \Phi G^* z, u) + \langle g(z,u), \nabla \Phi \rangle = 0,$$

относительно $\Phi(s,z) = \langle h, u(s,x) \rangle, \quad z = (x,h).$

$$TrG\nabla^2\Phi(s,x,h)G^* = A_{ik}\frac{\partial^2 \Phi(s,x,h)}{\partial x_i \partial x_j}A_{kj}$$

$$+2C_k^{lm} h_l \frac{\partial^2 \Phi(s,x,h)}{\partial x_j \partial h_m} A_{jk}, \quad \langle g, \nabla \Phi(s,x,h) \rangle$$

$$= a_j \frac{\partial \Phi(s,x,h)}{\partial x_j} + c_{lm} h_m \frac{\partial \Phi(s,x,h)}{\partial h_l}.$$

Shigesada, Kawasaki and Teramoto model of spatial segregation of interacting species (1979)

$$\frac{\partial u^m}{\partial t} = \Delta(u^m[u^1 + u^2]) + c_u^m u^m, \tag{19}$$

$$u^m(0,x) = u_0^m(x),\ m = 1,2,$$

where $c_u^m = c_m - c_{m1}u^1 - c_{m2}u^2$ and $c_m, c_{mk}, m, k = 1, 2$ are positive constants.

Keller-Segel model in chemotaxis

$$\frac{\partial u^m}{\partial t} = div(C^m(u^1,u^2)\nabla u^1 + B^m(u^1,u^2)\nabla u^2]) + c_u^m u^m, \tag{20}$$

$$u^m(0,x) = u_0^m(x),\ m = 1,2,$$

A pair of functions u^1, u^2 is a generalized solution of (19) if it has the following properties:

i) $u^1, u^2 \in L^\infty_{\rm loc}([0,\infty); L^\infty(R^d)) \cap \mathcal{W}$ and $u^1, u^2 \geq 0$ a.e. in $(0,\infty)\times R^d$;

ii) $\nabla u^m \in L^2_{\rm loc}((0,\infty)\times R^d)$,

iii) for any test function $h \in C_0^\infty([0,\infty)\times R^d)$ with compact support

$$\int_0^\infty \langle\langle u^m(\theta), [\frac{\partial h(\theta)}{\partial\theta} + [u^1(\theta)+u^2(\theta)]\Delta h(\theta)]\rangle\rangle d\theta$$

$$+\int_0^\infty \langle\langle u^m(\theta), c_u^m h(\theta)\rangle\rangle d\theta = -\langle\langle u_0^m, h(0)\rangle\rangle.$$

Set

$$\frac{1}{2}M_u^2(x) = u^1(t,x) + u^2(t,x),$$

$$c_u^m(x) = c_m - c_{m1}u^1(t,x) - c_{m2}u^2(t,x)$$

and consider the Cauchy problem for parabolic equations

$$\frac{\partial h^m(s,y)}{\partial s} + \frac{1}{2}M_u^2(y)\Delta h^m(s,y) + c_u^m(y)h^m(s,y) = 0, \tag{21}$$

$$h^m(t,y) = h^m(y), \quad 0 \le s \le t.$$

$$h^m(\theta, y) = E[\eta^m(t)h^m(\xi_{\theta,y}(t))], \quad 0 \le \theta \le t, m = 1, 2, \tag{22}$$

where $\xi(t), \eta^m(t)$ are governed by SDEs

$$d\xi(\theta) = M_u(\xi(\theta))dw(\theta), \quad \xi(0) = y, 0 \le \theta \le t, \tag{23}$$

$$d\eta^m(\theta) = c_u^m(\xi(\theta))\eta^m(\theta)d\theta, \quad \eta^m(0) = 1. \tag{24}$$

We construct a probabilistic representation of a regular generalized solution $u^m(t,x), m = 1,2$ of (19) assuming that $u^m(t,x)$ exists and unique. Under this assumption we can prove that there exists a unique solution $\xi(t)$ to (23) and its time reversal $\hat{\xi}(\theta)$ satisfies the stochastic integral equation

$$\hat{\xi}_{0,x}(\theta) = x - \int_\theta^t [M_u \nabla M_u](\hat{\xi}_{0,x}(\tau))d\tau - \int_\theta^t M_u(\hat{\xi}_{0,x}(\tau))d\tilde{w}(\tau), \qquad (24)$$

where $0 \le \theta \le \tau \le t$.

The main idea is the following

Let $\mathcal{W} = \{h \in C_0^{1,k}([0,T] \times R^d)\}$ be the space of test functions and $\mathcal{W}'$ be its dual. Define $u(t,\hat{\xi}(t))$ by

$$\int_0^{\mathbf{T}} \int_{\mathbf{R^d}} \mathbf{u(t,\hat{\xi}_{0,x}(t))h(t,x)dxdt}$$

$$= \int_0^{\mathbf{T}} \int_{\mathbf{R^d}} \mathbf{u(t,y)h(\xi_{0,y}(t)J_{0,t}(y)dydt}$$

where $J_{0,t}$ – Jacobian of the map $y = \hat{\xi}_{0,x}(t)$ Let $\eta(t)$ be a solution of

$$\mathbf{d}\eta(\theta) = \mathbf{c_u}(\xi_{\mathbf{0,y}}(\theta))\eta(\theta)\mathbf{d}\theta + \mathbf{C_u}(\xi_{\mathbf{0,y}}(\theta))\eta(\theta)\mathbf{dw}(\theta),$$

$\eta(0) = 1$ with continuous $c^u(x), C^u(x)$

Then

$$\int_0^T \int_{R^d} \tilde{\eta}(t) u(t, \hat{\xi}_{0,x}(t)) h(t, x) dx dt$$

$$= \int_0^T \int_{R^d} u(t, y) \eta(t) h(\xi_{0,y}(t) J_{0,t}(y) dy dt$$

where

$$\tilde{\eta}(t) = \exp \Big(\int_0^t [c_u(\hat{\xi}_{\theta,t}(x)) - \frac{1}{2} C_u^2(\hat{\xi}_{\theta,t}(x))] d\theta$$

$$+ \int_0^t C_u(\hat{\xi}_{\theta,t}(x)) dw(\theta) \Big).$$

Consider a stochastic process
$\gamma^m(\theta) = \eta^m(\theta) h(\xi(\theta)) J(\theta)$, where $\xi(\theta)$ satisfies (23) and the process $\eta^m(\theta)$ satisfies a linear SDE

$$d\eta^m(\theta) = \tilde{c}_u^m(\xi(\theta))\eta^m(\theta)d\theta + \eta^m(\theta)\langle C_u^m(\xi(\theta)), dw(\theta)\rangle, \tag{25}$$

with coefficients $\tilde{c}_u^m$ and C_u^m to be specified below.
In addition under suitable assumptions on functions u^m there exists $J(\theta) = det \nabla \xi_{0,x}(\theta)$. To obtain an explicit expression for $d\gamma^m(\theta)$ we apply the Ito formula and in addition derive an expression for a stochastic differential $dJ(\theta)$

$$dJ(\theta) = J(\theta)\langle \nabla M_u, dw(t)\rangle, \quad J(0) = 1. \tag{26}$$

Lemma

Let coefficients $\tilde{c}_u^m$ *and* C_u^m *in(18) have the form* $\tilde{c}_u^m(\xi(\theta)) = c_u^m(\xi(\theta)) - \langle \nabla M_u(\xi(\theta)), \nabla M_u(\xi(\theta)) \rangle$, $C_u^m(\xi(\theta)) = -\nabla M_u(\xi(\theta))$. *and* $\xi(\theta), \eta(\theta), J(\theta)$ *satisfy (17),(18),(19). Then the processes* $\gamma^m(\theta) = \eta^m(\theta) h(\xi_{0,y}(\theta)) J(\theta), m = 1, 2,$ *have stochastic differentials of the form*

$$d\gamma^m(\theta) = \left[\frac{1}{2} M_u^2 \Delta h + c_u^m h\right] (\xi(\theta)) \eta^m(\theta) J(\theta) d\theta$$

$$+ \langle M_u \nabla h(\xi(\theta)), \eta^m(\theta) J(\theta) dw(\theta) \rangle.$$

Next we consider a system

$$u^m(t,x) = E[\tilde{\eta}^m(t) u_0^m(\hat{\xi}_{0,x}(t))], \; m = 1, 2. \quad (27)$$

$$d\xi_{0,y}(\theta) = M_u(\xi_{0,y}(\theta)) dw(\theta), \quad \xi_{0,y}(0) = y, \quad (28)$$

$$d\eta^m(\theta) = \tilde{c}_u^m(\xi(\theta))\hat{\eta}^m(\theta) d\theta + C_u^m(\xi(\theta))\hat{\eta}^m(\theta) dw(\theta), \quad (29)$$

$$d\hat{\xi}_{0,x}(\theta) = [M_u \nabla M_u](\hat{\xi}_{0,x}(\theta)) d\theta + M_u(\hat{\xi}_{0,x}(\theta)) d\tilde{w}(\theta), \quad (30)$$

$$d\tilde{\eta}^m(\theta) = -\tilde{c}_u^m(\hat{\xi}(\theta))\tilde{\eta}^m(\theta) d\theta - C_u^m(\hat{\xi}(\theta))\tilde{\eta}^m(\theta) dw(\theta) + [C_u^m(\hat{\xi}(\theta))]^2 \tilde{\eta}^m(\theta) d\theta, \quad (31)$$

$$\eta^m(0) = 1, \quad \hat{\xi}_{0,x}(0) = x, \quad \tilde{\eta}^m(0) = 1.$$

Theorem

Assume that there exists a generalized solution $\mathbf{u}^1, \mathbf{u}^2$ *of the Cauchy problem (19). Then* $\mathbf{u}^m$ *admit a probabilistic representation of the form (27) where* $\hat{\xi}(\theta)$ *solves (30) and* $\tilde{\eta}(\theta)$ *solves (31) .*

Unfortunately we are not able still to treat the system (27),(30), (31) independent on the original system (19) since this system is not a closed one yet. To get a closed system we need a probabilistic representation for ∇u as well.

By formal differentiation of the system

$$\frac{\partial u^m}{\partial t} = \Delta[u^m(u^1+u^2)]+c_u^m u^m, \qquad u^m(0,x) = u_0^m(x), \tag{10}$$

we get a PDE for $v_i^m = \nabla_i u^m$

$$\frac{\partial v_i^m}{\partial t} = \Delta\{v_i^m(u^1+u^2)+u^m(v^1+v^2)\}+u^m\nabla_i c^m(u)$$
$$+c^m(u)v_i^m, \qquad v_i^m(0,x) = \nabla_i u_0^m(x). \tag{11}$$

$$\frac{\partial h}{\partial \theta} + (u^1 + u^2)\Delta h + c^m(u)h = 0, \quad h(T, y) = h_0(y), \tag{12}$$

we get a PDE for $g_i = \nabla_i h$

$$\frac{\partial g_i}{\partial \theta} + (u^1 + u^2)\Delta g_i + (v_i^1 + v_i^2) div g + \nabla_i c^m(u)h$$

$$+ c^m(u)g_i = 0, \quad g_i(0, y) = \nabla_i h_0(y). \tag{13}$$

In addition note that we can construct a stochastic representation of the solution to (12)-(13) in the form

$$G^m(t,x) = E[\eta^m(T)G_0(\xi_{\theta,x}(T))],$$

where $G(0,x) = (h_0(x), g_0(x))^*$ and stochastic processes $\xi(t)$ and $\eta_i^m(t)$ satisfy SDEs

$$d\xi(t) = \sqrt{2[u^1(t,\xi(t)) + u^2(t,\xi(t))}dw(t), \quad (14)$$

$$d\eta_j^m(t) = n_{j,u}^m(\xi(t)\eta_j(t)dt + N_{j,u}(\xi(t))\eta_j^m(t)dw(t).$$

Here δ is the Kronecker symbol,

$$n_{i,u}^m = \begin{pmatrix} c_u^m & 0 \\ \nabla_i c_u^m & c_u^m \end{pmatrix}, \quad N_{i,u}^m = \begin{pmatrix} 0 & 0 \\ 0 & \frac{[v_i^1+v_i^2]}{\sqrt{2(u^1+u^2)}}\delta \end{pmatrix} \quad (15)$$

$$G(\theta, x) = E\begin{pmatrix} \eta^m_{00}(T) & 0 \\ \eta^m_{i0}(T) & \eta^m_i(T) \end{pmatrix} \begin{pmatrix} h(\xi(T)) \\ \nabla h(\xi(T)) \end{pmatrix}, \quad (16)$$

and $\eta_i(T)\nabla h(\xi(T)) = \sum_{j=1}^d \eta_{ji}(T)\nabla_j h(\xi(T))$.

$$\frac{\partial}{\partial t}\begin{pmatrix} u^m \\ v^m_i \end{pmatrix} = \mathcal{Z}^m \begin{pmatrix} u^m \\ v^m_i \end{pmatrix}, \qquad m = 1, 2,\ i = 1, \ldots, d. \quad (17)$$

where setting $\bar{u} = u^1 + u^2, \bar{v}_i = v^1_i + v^2_i$ we obtain

$$\mathcal{Z}^m \begin{pmatrix} u^m \\ v^m_i \end{pmatrix} = \Delta\left[\begin{pmatrix} \bar{u} & 0 \\ \bar{v}_i & \bar{u} \end{pmatrix} \begin{pmatrix} u^m \\ v^m_i \end{pmatrix}\right] + \begin{pmatrix} c^m_{11} & 0 \\ c^m_{21} & c^m_{22} \end{pmatrix} \begin{pmatrix} u^m \\ v^m_i \end{pmatrix}. \quad (18)$$

Consider as well a dual system derived from above relations as follows. Integrate over R^d a product of (17) and a vector test function $(h, g_i)^*$, where $g_i = \nabla_i h$.

As a result we obtain a system of the form

$$\left\langle\left\langle \begin{pmatrix} u^m \\ v^m \end{pmatrix} \left[\frac{\partial}{\partial t} \begin{pmatrix} h \\ g \end{pmatrix} + \mathcal{Q}^m \begin{pmatrix} h \\ g \end{pmatrix} \right] \right\rangle\right\rangle = 0, \qquad (19)$$

where

$$\mathcal{Q}^m \begin{pmatrix} h \\ g \end{pmatrix} = \begin{pmatrix} \bar{u} & \bar{v} \\ 0 & \bar{u} \end{pmatrix} \Delta \begin{pmatrix} h \\ g \end{pmatrix} + \begin{pmatrix} c_{11}^m & 0 \\ c_{21}^m & c_{22}^m \end{pmatrix} \begin{pmatrix} h \\ g \end{pmatrix}$$

or in the form

$$\left\langle\left\langle \begin{pmatrix} u^m \\ v^m \end{pmatrix} \left[\frac{\partial}{\partial t} \begin{pmatrix} h \\ g \end{pmatrix} + \begin{pmatrix} \bar{u} & 0 \\ 0 & \bar{u} \end{pmatrix} \Delta \begin{pmatrix} h \\ g \end{pmatrix} \right] \right\rangle\right\rangle$$

$$+ \left\langle\left\langle \begin{pmatrix} u^m \\ v^m \end{pmatrix} \left[\begin{pmatrix} 0 & C_{12}^m \\ 0 & \tilde{0} \end{pmatrix} \begin{pmatrix} 0 \\ divg \end{pmatrix} + \begin{pmatrix} \tilde{c}_{11}^m & 0 \\ \tilde{c}_{21}^m & \tilde{c}_{22}^m \end{pmatrix} \begin{pmatrix} h \\ g \end{pmatrix} \right] \right\rangle\right\rangle$$

$$= 0.$$

Finally we get the closed system

$$d\hat{\xi}_{0,x}(\theta) = [M_u \nabla M_u](\hat{\xi}_{0,x}(\theta))d\theta + M_u(\hat{\xi}_{0,x}(\theta))d\tilde{w}(\theta),$$

$$d\hat{\zeta}^m(\theta) = \tilde{c}_u^m(\hat{\xi}(\theta))\hat{\zeta}^m(\theta)d\theta + C_u^m(\hat{\xi}(\theta)\hat{\zeta}^m(\theta)dw(\theta),$$

$$\hat{\xi}_{0,x}(0) = x, \quad \hat{\zeta}^m(0) = I$$

with coefficients c_u^m, $C_u^m = -\nabla M_u$ and

$$\begin{pmatrix} u^m(t,x) \\ \nabla u^m(t,x) \end{pmatrix} = E\left[\begin{pmatrix} \hat{\zeta}_{11}^m(\theta) & 0 \\ \hat{\zeta}_{21}^m(\theta) & \hat{\zeta}_{22}^m(\theta) \end{pmatrix} \begin{pmatrix} u_0^m(\hat{\xi}_{0,x}(\theta)) \\ v_0^m(\hat{\xi}_{0,x}(\theta)) \end{pmatrix}\right]$$

MHD-Burgers

$$\frac{\partial \mathbf{u}}{\partial \mathbf{t}} + \frac{\partial(\mathbf{uv})}{\partial \mathbf{x}} = \sigma^2 \frac{\partial \mathbf{u}^2}{\partial \mathbf{x}^2}, \quad \mathbf{u}(\mathbf{0}, \mathbf{x}) = \mathbf{u}_0(\mathbf{x}) \qquad (20)$$

$$\frac{\partial \mathbf{v}}{\partial \mathbf{t}} + \frac{\mathbf{1}}{\mathbf{2}} \frac{\partial(\mathbf{u}^2 + \mathbf{v}^2)}{\partial \mathbf{x}} = \mu^2 \frac{\partial \mathbf{v}^2}{\partial \mathbf{x}^2}, \quad \mathbf{v}(\mathbf{0}, \mathbf{x}) = \mathbf{v}_0(\mathbf{x}), \qquad (21)$$

We say $u(t,x) = (u_1(t,x), u_2(t,x))$ is a generalized solution of (20), (21) if

$$\sup_{t \in [0,T], x \in R} \|u(t,x)\|^2 < \infty$$

and $\forall h \in C_0^{1,\infty}[0, T) \times R, R^2)$ we have

$$\int_0^{\mathbf{T}} \langle\langle \mathbf{u_m}(\theta), \frac{\partial \mathbf{h_m}(\theta)}{\partial \theta} + (\mathcal{A}^{\mathbf{u}}_{\mathbf{m}} + \mathbf{B}^{\mathbf{u}}_{\mathbf{m}})\mathbf{h_k}(\theta)\rangle\rangle \, \mathbf{d}\theta$$

$$+\langle\langle \mathbf{u_{m0}}, \mathbf{h_m}(\mathbf{0})\rangle\rangle = \mathbf{0}, \qquad (22)$$

where $m = 1, 2,$

$$\mathcal{A}_1^u h_1 = \frac{1}{2}\sigma^2 \frac{\partial^2 h_1}{\partial x^2}, \qquad B_1^u h_1(x) = u_2 \frac{\partial h_1}{\partial x},$$

$$\mathcal{A}_2^u h_2 = \frac{1}{2}\mu^2 \frac{\partial^2 h_2}{\partial x^2}, \qquad B_2^u(x) h_2(x) = \left[\frac{u_1^2}{2u_2} + \frac{1}{2}u_2\right] \frac{\partial h_2}{\partial x}.$$

Consider SDE system

$$\mathbf{d}\xi^1(\theta) = \sigma\mathbf{dw}(\theta), \quad \xi^1(\mathbf{s}) = \mathbf{y}, \tag{23}$$

$$\mathbf{d}\xi^2(\theta) = \mu\mathbf{dw}(\theta), \quad \xi^2(\mathbf{s}) = \mathbf{y}, \tag{24}$$

$$\mathbf{d}\eta^1(\theta) = \mathbf{C}_1^{\mathbf{u}}(\xi^1(\theta))\eta^1(\theta)\mathbf{dw}(\theta), \quad \eta^1(\mathbf{s}) = \mathbf{1}, \tag{25}$$

$$\mathbf{d}\eta^2(\theta) = \mathbf{C}_2^{\mathbf{u}}(\xi^2(\theta))\eta^2(\theta)\mathbf{dw}(\theta), \quad \eta^2(\mathbf{s}) = \mathbf{1}, \tag{26}$$

where

$$\mathbf{C}_1^{\mathbf{u}}(\mathbf{x}) = \frac{\mathbf{1}}{\sigma}\mathbf{u}_2(\theta, \mathbf{x}), \quad \mathbf{C}_2^{\mathbf{u}}(\mathbf{x}) = \frac{\mathbf{1}}{\mathbf{2}\mu}\left[\mathbf{u}_2(\theta, \mathbf{x}) + \frac{\mathbf{u}_1^2(\theta, \mathbf{x})}{\mathbf{u}_2(\theta, \mathbf{x})}\right] \tag{27}$$

Stochastic test function in this case have the form $\gamma^m(\theta) = \eta^m(\theta)h(\xi^m(\theta))$ and

$$d\gamma^m(\theta) = \eta^m(\theta)[\mathcal{A}_u^m + B_u^m]h(\xi^m(\theta))d\theta \qquad (28)$$

$$+\eta^m(\theta)\left[C_m^u(\xi^m(\theta))h(\xi^m(\theta)) + \sigma_m\frac{\partial h(\xi^m(\theta))}{\partial y}\right]dw(\theta).$$

Let $\hat{\xi}^m(\theta)$ and $\tilde{\eta}^m(t)$ be given by

$$\mathbf{d\hat{\xi}^m}(\theta) = -\sigma_\mathbf{m}\mathbf{d\tilde{w}}(\theta), \qquad (28)$$

$$\tilde{\eta}^m(t) = exp\left\{\int_0^t C_m^u(\psi_{\theta,t}(x))dw(\theta)\right.$$

$$\left.-\frac{1}{2}\int_0^t [C_m^u]^2(\psi_{\theta,t}(x))d\theta\right\} \qquad (29)$$

Theorem

Assume that there exists a solution $u = (u^1, u^2) \in \mathcal{W} \times \mathcal{W}$ *of* (20), (21), $\psi^m_{0,\theta}(x) = \hat{\xi}^m_{0,x}(\theta)$, *satisfy (28), and* $\tilde{\eta}^m(t)$ *have the form (29) Then*

$$v^m(t) = \mathbf{E}\left[\tilde{\eta}_m(t) u_{0m} \circ \psi^m_{0,t}\right], \quad m = 1, 2, \tag{30}$$

satisfy integral identities

$$\frac{\partial}{\partial t} \int_R u_m(t,x) h(x) dx = \int_R [\mathcal{A}^u_m + \mathcal{B}^u_m] h(x) dx = 0. \tag{31}$$

Ladyzhenskaja O.A., Solonnikov V.A., Ural'tseva N.N. Linear and Quasi-linear Equations of Parabolic Type, AMS. 1988

Amann H. // Dynamic theory of quasilinear parabolic systems. III. Math. Z. 1989. V.202. N 2, C. 219–250.

H. Ishii, S.Koike (1991) Viscosity Solutions of a System of Nonlinear Second-Order Elliptic PDEs Arising in Switching Games. Funkcial. Ekvac., 34 (1991), pp. 143–155

D. H. Sattinger On the stability of waves of nonlinear parabolic systems. Advances of mathematics v. 22 (1976) 312 – 355

- Belopol'skaya, Ya., Daletskii Yu. Cauchy problem for a nonlinear parabolic system via operator-valued multiplicative functionals of Markov random processes Journal of Soviet Mathematics, Springer, 1983 v 21 N5, 653-657.
- Ya. Belopolskaya, Yu. Dalecky. Markov processes associated with nonlinear parabolic systems. DAN SSSR v. 250. N. 3, 1980 p.268-271.

- *Belopolskaya Ya., Woyczynski W.* // Generalized solution of the Cauchy problem for systems of nonlinear parabolic equations and diffusion processes, Stochastics and dynamics. 2012. V.11 N 1. P. 1–31.
- *Belopolskaya Ya.* // Probabilistic counterparts of nonlinear parabolic PDE systems, Modern Stochastics and Applications - Springer Optimization and Its Applications. 2014. V.90. 71–94.
- Belopolskaya,Ya.: Generalized solutions of nonlinear parabolic systems and vanishing

viscosity method. J.Math.Sci. **133** 1207–1223 (2006)

Belopolskaya, Ya., Woyczynski,W.: Generalized solution of the Cauchy problem for systems of nonlinear parabolic equations and diffusion processes. Stochastics and dynamics. **11**, 1, 1–31 (2012).

Kunita H.: Stochastic flows and stochastic differential equations.Cambridge Univ. Press, Cambridge (1990)

Kunita, H.: Stochastic flows acting on Schwartz distributions. J. Theor. Pobab. **7**, 2, 247-278 (1994)

- Kunita, H.: Generalized solutions of stochastic partial differential equations. J. Theor. Pobab. **7**, 2, 279–308 (1994)
- V. Bally, A. Matoussi. Weak Solutions for SPDEs and Backward Doubly Stochastic Differential Equations Journal of Theoretical Probability, Vol. 14, No. 1, 2001
- A. Matoussi, M. Xu Sobolev solution for semilinear PDE with obstacle under monotonicity condition Vol. 13 (2008), EJP. Paper no. 35, 1035–1067.

S. Peng, Probabilistic interpretation for systems of quasilinear parabolic partial differential equations. Stoch. and Stoch. Rep., 37 (1991) 61–74.

E. Pardoux. Backward stochastic differential equations and viscosity solutions of systems of semilinear parabolic and elliptic PDEs of second order. Stochastic Analysis and Relates Topics: The Geilo Workshop, Birkhäuser, (1996) 79–127.

- E. Pardoux, S. Peng, Backward stochastic differential equations and quasilinear parabolic partial differential equations, Lecture Notes in CIS, vol. 176, Springer, (1992) 200–217.
- M. Kac, Some stochastic problems in physics and mathematics, Magnolia Petroleum Co., Lectures in Pure and Applied Science No. 2 (1956).
- R. J. Griego, R. Hersch. Random evolutions, Markov chains and systems of partial differential equations Proc Nat. Acad Sci U S A. 1969 ; 62(2): 305–308.

M. Freidlin Functional integration and partial dierential equations Princeton Univ Press (1985)

E. Pardoux, F. Pradeilles, Z. Rao, Probabilistic interpretation of a system of semi-linear parabolic partial differential equations Annales de l'I.H.P, sec B, 33, 4 (1997) 467–490.