Gaussian functional inequalities and mass transportation on Wiener space

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LSA Winter meeting, Snegiri, 2017

Gaussian inequalities

Standard Gaussian measure

$$\gamma = \frac{1}{(2\pi)^{n/2}} e^{-\frac{|x|^2}{2}} dx$$

$g \cdot \gamma$ is another probability measure
Log-Sobolev inequality:

$$\mathrm{Ent}_\gamma(g) \leq \frac{1}{2}\mathrm{I}(g),$$

Talagrand transportation inequality:

$$\frac{1}{2} W_2^2(\gamma, g\cdot\gamma) \leq \mathrm{Ent}_\gamma(g).$$

$\mathrm{Ent}_\gamma(g) = \int g \log g d\gamma$ is the Gaussian **entropy**, $\mathrm{I}(g) = \int \frac{|\nabla g|^2}{g} d\gamma$ is the Gaussian Fisher **information**. $W_2(\mu, \nu)$ is the **Kantrorovich distance** between probability meaures μ, ν.

Gaussian inequalities in infinite-dimensional spaces

Gaussian measure on $\mathbb{R}^\infty$

$$\gamma = \prod_{i=1}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x_i^2}{2}} dx_i.$$

What is W_2?

$$W_2^2(\mu,\nu) = \inf_{P\in\Pi(\mu,\nu)} \int \sum_{i=1}^{\infty} (x_i - y_i)^2 dP.$$

Cost function is the square of the Cameron-Martin norm

$$c(x,y) = \sum_{i=1}^{\infty} (x_i - y_i)^2 = \|x-y\|^2.$$

Difficulty:

$$\|x-y\|^2 = +\infty \text{ for } \gamma - a.e. \text{ all } (x,y).$$

Infinite-dimensional transportation mapping

When does exist optimal transportation mapping?

$$T(x) = x + \nabla\varphi(x)$$

pushing forward μ onto ν such that

$$W_2^2(\mu, \nu) = \int \|T(x) - x\|^2 d\mu.$$

∇ is the Cameron-Martin gradient.

Existence of optimal transportation mapping

D. Feyel, A.S. Üstünel, 2004: Let

$$\mu = f \cdot \gamma,\ \nu = g \cdot \gamma$$

with $\mathrm{Ent}(f) < \infty, \mathrm{Ent}(g) < \infty$. Then there exists unique optimal transportation mapping T pushing forward μ onto ν.

K., 2003: Kullback-Leibler distance controls L_2 distance between optimal mappings:

Let T_f and T_h are optimal transportation mappings pushing forward $f \cdot \gamma$ and $h \cdot \gamma$ onto γ respectively. Then

$$\int f \log \frac{f}{h} d\gamma \geq \frac{1}{2} \int \| T_f - T_h \|^2 h d\gamma.$$

Infinite dimensional change of variables formula

$T(x) = x + \nabla\varphi(x)$ — infinite-dimensional optimal transportation

$$f(x) = g(T(x))e^{-\frac{1}{2}\|\nabla\varphi\|^2 + L\varphi}\det{}_2(I + D^2\varphi), \tag{1}$$

where

$$L\varphi = \sum_{i=1}^{\infty} \varphi_{x_i x_i} - x_i \varphi_{x_i},$$

$\det_2(I + D^2\varphi)$ is the corresponding Carleman-Fredholm determinant.

V.I. Bogachev, K. 2005, 2013: Partial results on validity of (1). In particular, (1) holds if $g = 1$, $\mathrm{I}(f) < \infty$.

Stability estimates

Stability for isoperimetric, Sobolev, Gagliardo-Nirenberg inequalities **A. Figalli, A. Pratelli, F. Maggi ...**

$$\frac{P(A)}{nVol^{1/n}(B_1)Vol(A)^{\frac{n-1}{n}}} - 1 \geq F(A),$$

$A \subset \mathbb{R}^n$, $P(A)$ - perimeter of ∂A, F is a non-negative function.
Stability for log-Sobolev and Talagrand inequalities
Fathi–Indrei–Ledoux 2014, Courtade–Fathi–Papanjady 2016, Bobkov–Gozlan–Roberto–Samson 2014.

$$\frac{1}{2}\mathrm{I}(g) - \mathrm{Ent}g \geq \delta_1(g),$$

$$\mathrm{Ent}g - \frac{1}{2}W_2^2(\gamma, g\cdot\gamma) \geq \delta_2(g),$$

Displacement convexity

Proofs of stability are refined arguments of displacement convexity arguments.
Displacement convex functionals: convex along geodesics on the space of measure endowed with Kantorovich metric.
Example. (Uniform) dispacement convexity of the entropy functional : $T(x) = x + \nabla\varphi(x)$ be the optimal transportation taking $g \cdot \gamma$ to $f \cdot \gamma$

$$\mathrm{Ent} f = \mathrm{Ent} g + \int \langle \nabla\varphi, \nabla g \rangle d\gamma + \frac{1}{2} W_2^2(g \cdot \gamma, f \cdot \gamma) + \int \left(\Delta\varphi - \log\det(I + D^2\varphi) \right) g d\gamma.$$

The last term is positive:

$$\mathrm{Ent} f \geq \mathrm{Ent} g + \int \langle \nabla\varphi, \nabla g \rangle d\gamma + \frac{1}{2} W_2^2(g \cdot \gamma, f \cdot \gamma).$$

Displacement convexity

$$\mathrm{Ent} f \geq \mathrm{Ent} g + \int \langle \nabla \varphi, \nabla g \rangle d\gamma + \frac{1}{2} W_2^2(g \cdot \gamma, f \cdot \gamma).$$

$g = 1$: Talagrand inequality; $f = 1$: Log-Sobolev inequality. Estimating $\Delta\varphi - \log\det(I + D^2\varphi)$ from below one can get refinements of these inequalities.

Bobkov–Gozlan–Roberto–Samson 2014:

$$\mathrm{Ent} g - \tfrac{1}{2} W_2^2(g \cdot \gamma, \gamma) \geq c \min\left\{ \tfrac{1}{2} n^{-1/2} W_{1,1}(g \cdot \gamma, \gamma), \tfrac{1}{4} n^{-1} W_{1,1}^2(g \cdot \gamma, \gamma) \right\},$$

where $W_{1,1}$ is the transportation cost for $c(x, y) = \sum_{i=1}^{n} |x_i - y_i|$.

K., 2012: log-Sobolev identity: let $f = 1$. Then

$$\mathrm{I}(g) = 2\mathrm{Entg} + F(\varphi),$$

where $F \geq 0$ is a functional defined on the potential φ.

Kähler–Einstein equation

$$\varrho(\nabla\Phi)\det D^2\Phi = e^{-\Phi},$$

where ϱ is a probaility density and Φ is a convex function.
Assumption: $\int x d\rho = 0$.
Well-posedness : **D. Cordero-Erausquin, B. Klartag, 2015**.
Approach: Φ is a maximum point of

$$J(f) = \log \int e^{-f^*} dx - \int f d\nu.$$

J is concave (Brunn-Minkowsli inequality)
F. Santambrogio, 2015 $\rho = e^{-\Phi}$ gives a minimum to the functional

$$\mathcal{F}(\rho) = -\frac{1}{2} W_2^2(\nu, \rho dx) + \frac{1}{2}\int x^2 \rho \; dx + \int \rho \log \rho dx. \qquad (2)$$

$\mathcal{F}$ is not convex, but displacement convex.

Kähler–Einstein equation on Wiener space

$$n = \infty$$

Following Santambrogio consider the following functional

$$\mathcal{F}_\gamma(\rho) = -\frac{1}{2} W_2^2(g \cdot \gamma, \rho \cdot \gamma) + \int \rho \log \rho \; d\gamma,$$

where $g \cdot \gamma, \rho \cdot \gamma$ are probability measures.
Problem: Does $\mathcal{F}_\gamma$ admits a (unique) minimum point?

Theorem

K., E. Kosov *Assume that* g *is a probability density satisfying* $I(g) < \infty$, *and* $\int x_i g d\gamma = 0$ *for every* i. *Then there exists a constant* $C > 0$ *depending only on* $I(g)$ *such that*

$$\mathcal{F}_\gamma \geq -C.$$

Important ingredients of the proof:
1) stability estimate of Fathi–Indrei–Ledoux: assume that $f \cdot \gamma$ satisfies the Poincaré inequality with constant c and $\int xfd\gamma = 0$. Then

$$\Big(\frac{1}{2} - \varepsilon(c)\Big)\mathrm{I}(f) \geq \mathrm{Ent} f.$$

2) Concentration–isoperimetry equivalence result of E. Milman.