ASYMPTOTIC PROPERTIES OF DIFFUSION TYPE SEMIGROUPS

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- Wiener path integral representation for heat kernel

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- Application to regularized heat trace estimation

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- Schwartz kernel short-time asymptotics

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heat invariants $a_j(V)$ – KdV first integrals $(d = 1)$

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$$L_x(\varphi) \;:=\; \int \dots \int F(x_1, \dots, x_m)\,p_0(x, x_1, t_1) \times$$

$$\times\, p_0(x_1, x_2, t_2 - t_1) \dots p_0(x_{m-1}, x_m, t_m - t_{m-1})\,dx_1 \dots dx_m$$

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Extension $L_x(\varphi) = \int_{\Omega_x} \varphi(\omega)\, d\mu_x(\omega), \quad \mu_x(\Omega_x) = 1$

$$\int_{\Omega_x} f(\omega(t))\, d\mu_x(\omega) = \int f(y) p_0(x,y,t)\, dy \ = \ U_0(t) f(x)$$

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Duhamel equation $\;\mapsto\;$ Perturbation theory expansion

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$$\int_0^t ds \int_{\Omega_x} f(\omega(t)) V(\omega(s))\, d\mu_x(\omega) \;=$$

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Short-time asymptotics

$$p_V(x,y,t) \sim p_0(x,y,t) \left\{1 + \sum_{n=1}^{\infty} c_n(x,y) t^n\right\},\ t \downarrow 0$$

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Gaussian measure momenta

$$(2\pi)^{-m/2}(\det a_{ij})^{-1/2} \int \dots \int x_1^{k_1} \dots x_m^{k_m} \exp\left(-\frac{a^{ij}}{2} x_i x_j\right) dx_1 \dots dx_m$$

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$=: E_k(s_1, \dots, s_m),\ \ k = (k_1, \dots k_m),\ \ a_{ij} = \min\{s_i, s_j\} - s_i s_j$

Theorem 1 *Given complex-valued bounded* $V(x) \in C^\infty(\mathbb{R}^3)$

$$c_n(x,y) = \sum_{m=1}^{n} \sum_{\substack{\alpha+\beta+\gamma=k \\ |k|=2(n-m)}} \int_0^1 \partial_{x_1}^{\alpha_1} \partial_{x_2}^{\beta_1} \partial_{x_3}^{\gamma_1} V(\xi(s_1))\, ds_1 \times$$

$$\int_0^{s_1} \partial_{x_1}^{\alpha_2} \partial_{x_2}^{\beta_2} \partial_{x_3}^{\gamma_2} V(\xi(s_2))\, ds_2 \dots \int_0^{s_{m-1}} \partial_{x_1}^{\alpha_m} \partial_{x_2}^{\beta_m} \partial_{x_3}^{\gamma_m} V(\xi(s_m)) \times$$

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$$\Phi_{\alpha\beta\gamma}(s_1, \ldots s_m) \ := \left(\prod_{j=1}^m \frac{1}{\alpha_j! \beta_j! \gamma_j!} \right) E_\alpha(s_1, \ldots, s_m) E_\beta(s_1, \ldots, s_m) E_\gamma(s_1, \ldots, s_m)$$

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Corollary $c_n(x,y)$ *is homogeneous in* V *of degree* $= n$ *if each* ∂_x *is counted with wieght 1/2*

Heat invariants

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$$+\int_0^1 \left\langle \nabla V(\xi(t))\int_0^t \nabla V(\xi(s))sds\right\rangle (1-t)dt$$

$$c_4(x,y) = \frac{1}{48}\int_0^1 \Delta^3 V(\xi(s))(s(1-s))^3 ds + \frac{1}{24}\left(\int_0^1 V(\xi(s))\, ds\right)^4 +$$

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$$+ \frac{1}{2}\left\{ \int_0^1 \left\langle \nabla(\Delta V)(\xi(t)) \int_0^t \nabla V(\xi(s))\, s ds \right\rangle t(1-t)^2 dt\right.$$

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$$\left. + \int_0^1 \left\langle \nabla V(\xi(s)) \int_0^s \nabla(\Delta V)(\xi(t))\, t^2(1-t) dt \right\rangle (1-s) ds \right\}$$

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\end{aligned}
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$$c_4(x,y) = \frac{1}{48}\int_0^1 \Delta^3 V(\xi(s))(s(1-s))^3 ds + \frac{1}{24}\left(\int_0^1 V(\xi(s))\, ds\right)^4 +$$
$$\frac{1}{8}\left\{\int_0^1 \Delta^2 V(\xi(s))(s(1-s))^2 ds \int_0^1 V(\xi(s)) ds + \left(\int_0^1 \Delta V(\xi(s)) s(1-s) ds\right)^2\right\}$$
$$+ \frac{1}{2}\left\{ \int_0^1 \left\langle \nabla(\Delta V)(\xi(t)) \int_0^t \nabla V(\xi(s))\, s ds \right\rangle t(1-t)^2 dt\right.$$
$$\left. + \int_0^1 \left\langle \nabla V(\xi(s)) \int_0^s \nabla(\Delta V)(\xi(t))\, t^2(1-t) dt \right\rangle (1-s) ds \right\}$$
$$+ \frac{1}{4}\left(\int_0^1 V(\xi(s))\, ds\right)^2 \int_0^1 \Delta V(\xi(s))\, s(1-s)) ds$$
$$+ \int_0^1 V(\xi(s))\, ds \int_0^1 \left\langle \nabla V(\xi(t)) \int_0^t \nabla V(\xi(s))\, s ds \right\rangle (1-t) dt$$
$$+ \frac{1}{2}\int_0^1 (1-t)^2 dt \int_0^t \mathrm{Tr}\left\{ V''_{xx}(\xi(t)) \cdot V''_{xx}(\xi(s))\right\} s^2 ds$$

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$$\geqslant \exp \int_0^t \underbrace{\left(\int_{\Omega^t_{x,x}} V(\omega(s))\, d\mu^t_{x,x}(\omega) \right)}_{\left(2\pi \frac{s}{t}\left(1-\frac{s}{t}\right)\right)^{-3/2} \int V(x+\sqrt{t}\xi) \exp\left(-|\xi|^2/2\frac{s}{t}\left(1-\frac{s}{t}\right)\right) d\xi} ds$$

Theorem 2 *Given continuous bounded potential* $V \in \mathrm{L}_1(\mathbb{R}^3)$

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(M. Zworski & A. Sa Barreta, 1996) For negative $V \in \mathrm{L}_1(\mathbb{R}^3)$

$$\frac{1}{2} \int_{|V(x)|<1/t} |V(x)|\, dx \leqslant (2\pi)^{3/2} t^{1/2} \mathrm{Tr}(U_0(t) - U(t)) \leqslant \int |V(x)|\, dx$$

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$$\left.\left. -\langle \nabla\times a\,(\xi(s))(y-x)\rangle\langle \nabla\times a\,(\xi(r))(y-x)\rangle\Big]\,r\,dr\right) + O(t^{5/4})\right\}$$

Outline of the proof Conditional Wiener measure $\mu^t_{x,y}$ is supported on Brownian paths

$$\omega(s) = \left(1 - \frac{s}{t}\right) x + \frac{s}{t} y + \sqrt{t}\, b\left(\frac{s}{t}\right), \; s \in [0, t]$$

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$$C_0 \;=\; 1/4\,,\quad C_1 \;=\; (2\pi)^{-3}\int e^{-P(\xi)/10}\,d\xi$$

$$G_0(x-y,t) \sim \frac{2}{\sqrt{3}} \frac{(2\pi)^{-3/2}}{|x-y|\sqrt{t}} \times$$

$$\mathrm{Im}\left\{ \exp\left(\frac{3}{4}\frac{|x-y|^{4/3}}{t^{1/3}}\, e^{-2\pi i/3}\right)\left(1+\sum_{k=1}^{\infty} a_k \left(\frac{t^{1/3}}{|x-y|^{4/3}}\right)^k\right)\right\}$$

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$$\left\{ \sin\left(\frac{3\sqrt{3}}{8} \frac{|x-y|^{4/3}}{t^{1/3}} \right) - \frac{5}{36} \sin\left(\frac{3\sqrt{3}}{8} \frac{|x-y|^{4/3}}{t^{1/3}} - \frac{2\pi}{3} \right) \frac{t^{1/3}}{|x-y|^{4/3}} + O\left(\frac{t^{2/3}}{|x-y|^{8/3}} \right) \right\}, \quad t \downarrow 0$$

Path integral representation is replaced by **Parametrix**

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$$\forall\,\alpha \in (1,2] \;\; \exists\, C(\alpha) > 0 \;\; \forall\, a,b \in \mathbb{R}^3 \;\; \forall\,\tau \in (0,1)$$

$$\frac{|a\tau + b|^\alpha}{\tau^{\alpha-1}} + \frac{|a(1-\tau) - b|^\alpha}{(1-\tau)^{\alpha-1}} \;\geqslant\; |a|^\alpha + C(\alpha)|b|^2 \max\{|a|,|b|\}^{\alpha-2}$$

$$\sum_{n>2} G^{(n)}(x,y,t) = O\big(t\, p(x,y,t)\big),\ p(x,y,t) = \exp\left(-\frac{3}{8}\frac{|x-y|^{4/3}}{t^{1/3}}\right)$$

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Born approximation

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analogue of mathematical expectation $\mu(s) = x + (y-x)s/t$

$$\int G_0(x-z,s)\, z\, G_0(z-y,t-s)\, dz = \int \exp\left(-sP(\xi)+i\langle x,\xi\rangle\right) d\xi$$

$$\times \int \exp\left((s-t)P(\eta)-i\langle y,\eta\rangle\right) d\eta\ (2\pi)^{-6} \underbrace{\int z e^{i\langle z,\eta-\xi\rangle} dz}_{(2\pi)^3\left(-i\nabla\delta_\xi(\eta)\right)} =$$

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$$\left\langle \int \exp\left((s-t)P(\eta)-i\langle y,\eta\rangle\right)\nabla\delta_\xi(\eta)\,d\eta = \right.$$

$$\left. = \exp\left((s-t)P(\xi)-i\langle y,\xi\rangle\right)\left((t-s)\nabla P(\xi)+iy\right)\right\rangle$$

$$= -i\,(2\pi)^{-3}\int \exp\left(-tP(\xi)+i\langle x-y,\xi\rangle\right)\left(iy+(t-s)\nabla P(\xi)\right)d\xi$$

$$= y\,G_0(x-y,t) + (t-s)\frac{x-y}{t}\,G_0(x-y,t) = G_0(x-y,t)\mu(s)$$

$$\Big\langle \int \exp\big(- tP(\xi) + i\,\langle x-y,\xi\rangle\big)\nabla P(\xi)\,d\xi =$$

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Mean deviation $(V(x)\in \mathrm{C}^2(\mathbb{R}^3))$

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Theorem 4 $V(x) \in \mathrm{C}^2(\mathbb{R}^3) \cap \mathrm{L}_1(\mathbb{R}^3)$ *bounded potential*

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off-diagonal short-time asymptotics

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$$= t\, G_0(0,t) \int V(z)\,dz, \quad G_0(0,t) = (2\pi)^{-3} t^{-3/4} \int e^{-P(\xi)} d\xi$$