Quantifying non-monotonicity of functions

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Introduction.

In various research areas related to decision making, problems and their solutions frequently rely on certain functions being monotonic. In the case of non-monotonic functions, one would then wish to quantify their lack of monotonicity. In this talk we discuss a method designed specifically for this task, including quantification of the lack of positivity, negativity, or sign-constancy in signed measures.

Problem.

Let

$$f, g : [0, 1] \to \mathbb{R}^1.$$

Which of them is more monotone (more increasing) ?

Example.
Consider the family $\{\phi_t\}$, $t \in (0, 1)$, of functions :

$$\phi_t(x) = x/t, \quad x \in [0, t] \quad \textit{and} \quad \phi_t(x) = (1 - x)/(1 - t), \quad x \in [t, 1].$$

Approach.

Suppose f, g to be absolutely continuous, $f(0) = g(0) = 0$. Then it seams reasonable to think that f is more increasing than g if f' is more positive than g'.

Hence we can formulate the problem in the following way. Let

$$K_+ = \{h \,|\, h \geq 0 \; a.e.\}, \;\; K_- = -K_+,$$

and d be a metric on the space of integrable functions.

Definition

We say that f' is more positive than g' if $d(f', K_+) \leq d(g', K_+)$

First result.

Let $f, g \in K_+$. Denote $d_p(g,h) = \|g-h\|_p$, $1 \le p \le \infty$.
The choice $p = 1$ seams to be the most adequate.

Proposition 1.

Let $\Delta_p^+(f) = d_p(f,\, K_+)$, $\Delta_p^-(f) = d_p(f,\, K_-)$ and

$$A_+ = \{g \in K_+ \mid \|f-g\| = \Delta_p^+(f)\},$$

$$A_- = \{g \in K_- \mid \|f-g\| = \Delta_p^-(f)\},$$

Then $\Delta_p^+(f) = \|f_-\|_p$, $\Delta_p^-(f) = \|f_+\|_p$, and $A_+ = \{f_+\}$, $A_- = \{f_-\}$.

Rem. 1. The sets A_+, A_- are the same independently on p.

Rem. 2. The decomposition $f = f_+ - f_-$ is minimal in the sense that if $f = g - h$ with $g, h \in K_+$ is another decomposition, then $g \ge f_+$, $h \ge f_-$ a.s.

Example.

Let us come back to our example. We have

$\Delta_1^+(\phi_t) = 1 \quad \text{for all } t; \quad \Delta_1^-(\phi_t) = 1.$

$\Delta_p^+(\phi_t) = (1-t)^{-\frac{p-1}{p}} \quad \text{for } 1 < p < \infty, \quad \Delta_p^-(\phi_t) = t^{-\frac{p-1}{p}}.$

$\Delta_\infty^+(\phi_t) = \frac{1}{1-t}, \quad \Delta_\infty^-(\phi_t) = \frac{1}{t}.$

Index of monotonicity.

The quantity

$$I(f) = 1 - 2\min\left\{\frac{\Delta_1^+(f)}{\|f\|_1}, \frac{\Delta_1^-(f)}{\|f\|_1}\right\}$$

can be considered as an index of monotonicity :

1) $0 \leq I(f) \leq 1$;

2) $I(cf) = I(f) \quad \text{for} \quad c > 0$;

3) $I(f) = 1 \quad \text{iff} \quad f \in K_+ \quad \text{or} \quad f \in K.$

Functions of bounded variation.

Function f has a bounded variation iff $f = g - h$, where g, h are nondecreasing functions. The signed measure μ corresponding to f is a difference of two positive measures

$$\mu = \nu - \rho. \tag{1}$$

This representation is not unique. The minimal one is given by by the Jordan decomposition $\mu = \mu_+ - \mu_-$, where

$$\mu_+ = \frac{1}{2}(|\mu| + \mu), \quad \mu_- = \frac{1}{2}(|\mu| - \mu),$$

$|\mu|$ being the total variation measure associated to μ.

Functions of bounded variation.

Proposition 2

Let $\|\mu\|$ *be the total variation norm,*
M_+ *be the set of all finite positive measures,*
$\Delta_+(\mu) = dist(\mu, M_+)$ *and* $A = \{\nu \mid dist(\nu, M_+) = \Delta_+(\mu)\}$.

Then

$$\Delta_+(\mu) = \|\mu_-\|, \quad A = \{\mu_+\}.$$

Corollary

(Proposition 1, $p = 1$)
In this case $\mu \ll \lambda$; $\frac{d\mu}{d\lambda} = h$; $\frac{d\mu_+}{d\lambda} = h_+$, $\frac{d\mu_-}{d\lambda} = h_-$,
and we have

$$\|\mu_-\| = \|h_-\|_1.$$

Proof of Proposition 2

It is clear that $\Delta_+(\mu) \le \|\mu_-\|$.

Let E, F be a Hahn decomposition for μ. It means that $E \cap F = \emptyset$, $\;E \cup F = [0,1]$ and for each $B \subset [0,1] \quad \mu_+(B) = \mu(B \cap E), \quad \mu_-(B) = \mu(B \cap F)$.

Let $\nu \in M_+$. We have

$$\begin{aligned}
&\|\mu - \nu\| = \|\mu_+ - \mu_- - \nu\| = \\
&= |\mu_+ - (\mu_- + \nu)|(E) + |\mu_+ - (\mu_- + \nu)|(F) \ge \\
&\ge |\mu_+ - (\mu_- + \nu)|(F) = |\mu_- + \nu|(F) \ge \\
&\ge |\mu_-|(F) = \|\mu_-\|.
\end{aligned}$$

Hence $\Delta_+(\mu) \ge \|\mu_-\|$.

General setting

Let B be a Banach space, K be a closed cone in B such that $0 \in K$ and $K - K = B$, that is $\forall x \in B \ \exists \ y \in K, \ z \in K$ s.th. $x = y - z$. We say that $x \geq 0$ if $x \in K$ and we say $x \geq y$ if $y - x \geq 0$.

Questions :

1) When there exist a unic "minimal" decomposition of x in difference of two "positive" elements ? More exactly, under what conditions for each $x \in B \quad \exists \,! \ \ y_0, z_0 \in K$ such that $x = y_0 - z_0$ and from another equality $x = y - z, \ \ y, z \in K$ it follows that $y_0 \leq y, \ \ z_0 \leq z$?

2) Under what conditions on B and on metric d $\inf\{d(x, K)\}$ is reached in y_0 (respectively, $\inf\{d(x, -K)\}$ is reached in z_0) ?

Comments

1. Relation between "to be more increasing" and "to be more positive".
2. Distance $d_p(f, g) = \|f_g\|_p$ for $p \in (0, 1)$.
3. Banach space with strictly convex norme.

References

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