Generalized Space-Time Fractional Equation and the Related Stochastic Processes

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Moscow
24-28 July 2017

In this talk we consider the following space-time fractional equation

$$\sum_{j=1}^{m} \lambda_j \frac{\partial^{\nu_j}}{\partial t^{\nu_j}} w(x_1, \dots, x_n; t) = -c^2 (-\Delta)^{\beta} w(x_1, \dots, x_n, t) \qquad (1)$$

subject to the initial condition

$$w(x_1, \dots, x_n; 0) = \prod_{j=1}^{m} \delta(x_j) \qquad (2)$$

and where $0 < \nu_j < 1$, $0 < \beta \leq 1$.

The time-fractional derivatives must be understood in the sense of Dzerbayshan-Caputo and, in our case, writes

$$\frac{\partial^{\nu_j}}{\partial t^{\nu_j}} w(x_1, \ldots, x_n; t) = \frac{1}{\Gamma(1-\nu_j)} \int_0^t \frac{\frac{\partial}{\partial s} w(x_1, \ldots, x_n; s)}{(t-s)^{\nu_j}} \, ds \qquad (3)$$

The fractional Laplacian appearing in (1) is defined in terms of Fourier transforms for a function $u(\vec{x}) = u(x_1, \ldots, x_n)$ as

$$-(-\Delta)^{\beta} u(\vec{x}) = -\frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{-i\vec{x}\cdot\vec{\xi}} \, \|\vec{\xi}\|^{2\beta} \, \hat{u}(\vec{\xi}) \, d\vec{\xi} \qquad (4)$$

where $\hat{u}(\vec{\xi})$ is the Fourier transform of $u(\vec{x})$.

Equation (1) includes as special cases the following time-fractional one-dimensional telegraph equation

$$\left(\frac{\partial^{2\nu}}{\partial t^{2\nu}} + 2\lambda \frac{\partial^{\nu}}{\partial t^{\nu}}\right) u(x,t) = c^2 \frac{\partial^2}{\partial x^2} u(x,t), \quad x \in \mathbb{R},\ t > 0,\ 0 < \nu \le 1 \tag{5}$$

which itself generalises the telegraph equation ($\nu = 1$) which is the governing equation of the distribution of the telegraph process.

We have been able to write down the Fourier transform $u(\xi, t)$ of the solution of (5) subject to the initial conditions

$$u(x,0) = \delta(x) \qquad 0 < \nu \leq \frac{1}{2}$$

$$u(x,0) = 0 \qquad \frac{1}{2} < \nu \leq 1$$

as

$$u(\xi, t) = \frac{1}{2}\left[\left(1 + \frac{\lambda}{\sqrt{\lambda^2 - c^2\xi^2}}\right) E_{\nu,1}(r_1 t^\nu) + \left(1 - \frac{\lambda}{\sqrt{\lambda^2 - c^2\xi^2}}\right) E_{\nu,1}(r_2 t^\nu)\right] \qquad (6)$$

where $r_1 = -\lambda + \sqrt{\lambda^2 - c^2\xi^2}$, $r_2 = -\lambda - \sqrt{\lambda^2 - c^2\xi^2}$ and $E_{\nu,1}(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\nu k+1)}$ is the one-parameter Mittag-Leffler function.

A special and interesting subcase of (5) is when $\nu = \frac{1}{2}$, for which (6) can be inverted explicitly and coincides with the distribution of

$$T(|B(t)|)$$

where T is a telegraph process independent from $|B(t)|$, which is a reflecting Brownian motion.

For $\lambda \to \infty$, $c \to \infty$ in such a way that $\frac{c^2}{\lambda} \to 1$, equation (5) becomes

$$\frac{\partial^\nu}{\partial t^\nu} u(x,t) = \frac{\partial^2}{\partial x^2} u(x,t) \tag{7}$$

and for $\nu = \frac{1}{2}$ its fundamental solution coincides with the distribution of the iterated Brownian motion

$$I(t) = B_1\left(|B_2(t)|\right) \tag{8}$$

$B_j(t)$, $j = 1, 2$ being independent Brownian motions.

By the way, the distribution of (8) coincides with the fundamental solution of the non-homogeneous fourth-order equation

$$\frac{\partial}{\partial t}u(x,t) = \frac{1}{2^3}\frac{\partial^4}{\partial x^4}u(x,t) + \frac{1}{2\sqrt{2\pi t}}\frac{d^2}{dx^2}\delta(x) \tag{9}$$

where $\delta(x)$ is the Dirac delta function.

We now return to the general equation (1), of which we are able to give a probabilistic solution as the distribution of a time-changed isotropic stable process.
Let us now formulate this result explicitly.

The solution of the Cauchy problem

$$\begin{cases} \sum_{j=1}^{m} \lambda_j \frac{\partial^{\nu_j}}{\partial t^{\nu_j}} w(x_1, \ldots, x_n; t) = -c^2 (-\Delta)^{\beta} w(x_1, \ldots, x_n; t) \\ w(x_1, \ldots, x_n; 0) = \delta(x_1, \ldots, x_n) = \prod_{j=1}^{m} \delta(x_j) \end{cases} \tag{10}$$

for $0 < \nu_j \leq 1$, $0 < \beta \leq 1$ coincides with the distribution of the process

$$W_n^{\nu_1, \ldots, \nu_n}(t) = S_n^{2\beta} \left(c^2 \mathcal{L}^{\nu_1, \ldots, \nu_m}(t) \right) \tag{11}$$

where $S_n^{2\beta}$ is an isotropic stable process and $\mathcal{L}^{\nu_1, \ldots, \nu_m}(t)$ is now defined as the inverse of a suitable combination of stable subordinators.

The process $S_n^{2\beta}(t)$ has distribution

$$v_\beta(\vec{x}, t) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{-i\vec{\xi}\cdot\vec{x}} e^{-t\|\xi\|^{2\beta}} d\vec{\xi} \tag{12}$$

and therefore has characteristic function

$$\hat{v}_\beta(\vec{\xi}, t) = e^{-t\|\xi\|^{2\beta}}$$

A bit more complicated is the definition of $\mathcal{L}^{\nu_1, \dots, \nu_m}(t)$.

We first consider

$$\mathcal{H}^{\nu_1,\dots,\nu_m}(t) = \sum_{j=1}^{m} \lambda_j^{\frac{1}{\nu_j}} H_j^{\nu_j}(t), \quad 0 < \nu_j < 1 \tag{13}$$

where $H_j^{\nu_j}(t)$ are independent, stable subordinators of order $0 < \nu_j < 1$.

The process $\mathcal{L}^{\nu_1,\dots,\nu_m}(t)$ is the inverse of $\mathcal{H}^{\nu_1,\dots,\nu_m}(t)$ and is defined as

$$\mathcal{L}^{\nu_1,\dots,\nu_m}(t) = \inf\left(s > 0 : \mathcal{H}^{\nu_1,\dots,\nu_m}(t) = \sum_{j=1}^{m} \lambda_j^{\frac{1}{\nu_j}} H_j^{\nu_j}(s) \geq t \right) \tag{14}$$

The distribution of $\mathcal{H}^{\nu_1,\dots,\nu_m}(t)$ and $\mathcal{L}^{\nu_1,\dots,\nu_m}(t)$ are related as

$$\Pr\left\{\mathcal{L}^{\nu_1,\dots,\nu_m}(t) < x\right\} = \Pr\left\{\mathcal{H}^{\nu_1,\dots,\nu_m}(x) > t\right\} \tag{15}$$

As far as the distributions $\ell_{\nu_1, \dots, \nu_m}(t)$ of $\mathcal{L}^{\nu_1, \cdots, \nu_m}(t)$ and $h_{\nu_1, \dots, \nu_m}(t)$ of $\mathcal{H}^{\nu_1, \cdots, \nu_m}(t)$ we have the following theorem.

Theorem

(i) For $x > 0$, $t > 0$ and $0 < \nu_j < 1$, the solution to the problem

$$\begin{cases} \frac{\partial}{\partial t} h_{\nu_1, \dots, \nu_m}(x,t) = -\sum_{j=1}^{m} \lambda_j \frac{\partial^{\nu_j}}{\partial x^{\nu_j}} h_{\nu_1, \dots, \nu_m}(x,t) \\ h_{\nu_1, \dots, \nu_m}(x,0) = \delta(x) \\ h_{\nu_1, \dots, \nu_m}(0,t) = 0 \end{cases}$$

is given by the density of $\mathcal{H}^{\nu_1, \cdots, \nu_m}(t)$, $t > 0$.

Theorem

(ii) For $x > 0$, $t > 0$, the solution to the problem

$$\begin{cases} \sum_{j=1}^{m} \lambda_j \frac{\partial^{\nu_j}}{\partial t^{\nu_j}} \ell_{\nu_1,\dots,\nu_m}(x,t) = -\frac{\partial}{\partial x} \ell_{\nu_1,\dots,\nu_m}(x,t) \\ \ell_{\nu_1,\dots,\nu_m}(0,t) = \sum_{j=1}^{m} \lambda_j \frac{t^{-\nu_j}}{\Gamma(1-\nu_j)} \end{cases}$$

is given by the density of $\mathcal{L}^{\nu_1,\dots,\nu_m}(t)$, $t > 0$.

The fractional derivatives appearing above must be understood in the Riemann-Liouville sense.

The technique used for the proof of both statements is based on Laplace transforms in case (ii) and Fourier transforms in case (i). The combination of Fourier-Laplace transforms is the key tool for proving the main statement about the solution of the Cauchy problem (10), which produces for

$$\hat{\hat{w}}_{\nu_1, \ldots, \nu_n}(\xi_1, \ldots, \xi_n; \mu) = \hat{\hat{w}}(\vec{\xi}, \mu) = \int_0^\infty e^{-\mu t} dt \int_{\mathbb{R}^n} e^{i\vec{\xi}\cdot\vec{x}} w(\vec{x}, t)\, d\vec{x}$$

$$\hat{\hat{w}}(\vec{\xi}, \mu) = \frac{\sum_{j=1}^m \lambda_j\, \mu^{\nu_j - 1}}{\sum_{j=1}^m \lambda_j\, \mu^{\nu_j} + c^2 \|\xi\|^{2\beta}} \tag{16}$$

If now we take the Fourier-Laplace transform of the process

$$S_n^{2\beta}\left(c^2\mathcal{L}^{\nu_1,\,\dots,\,\nu_m}(t)\right) \tag{17}$$

this check can be by first evaluating the characteristic function of the process (17) as

$$\begin{aligned}
&\mathbb{E}\left[e^{i\vec{\xi}\cdot S_n^{2\beta}\left(c^2\mathcal{L}^{\nu_1,\,\dots,\,\nu_m}(t)\right)}\right]\\
=&\,\mathbb{E}\left[\mathbb{E}\left(e^{i\vec{\xi}\cdot S_n^{2\beta}\left(c^2\mathcal{L}^{\nu_1,\,\dots,\,\nu_m}(t)\right)}\mid\mathcal{L}^{\nu_1,\,\dots,\,\nu_m}(t)\right)\right]\\
=&\,\mathbb{E}\left[e^{-c^2\|\xi\|^{2\beta}\mathcal{L}^{\nu_1,\,\dots,\,\nu_m}(t)}\right]\\
=&\int_0^\infty e^{-c^2 s\|\xi\|^{2\beta}}\ell_{\nu_1,\,\dots,\,\nu_m}(s,t)\,ds
\end{aligned} \tag{18}$$

Consider now

$$\Pr\left\{\mathcal{L}^{\nu_1,\,\dots,\,\nu_m}(t) < s\right\} = \Pr\left\{\mathcal{H}^{\nu_1,\,\dots,\,\nu_m}(s) > t\right\} = \int_t^\infty \Pr\left\{\mathcal{H}^{\nu_1,\,\dots,\,\nu_m}(s) \in dz\right\} = \int_t^\infty h_{\nu_1,\,\dots,\,\nu_m}(z,s)\,dz \tag{19}$$

so that

$$\ell_{\nu_1,\,\dots,\,\nu_m}(s,t) = -\frac{\partial}{\partial s}\int_0^t h_{\nu_1,\,\dots,\,\nu_m}(z,s)\,dz$$

By plugging (19) into (18) we get

$$\mathbb{E}\left[e^{i\vec{\xi}\cdot S_n^{2\beta}\left(c^2\mathcal{L}^{\nu_1,\,\dots,\,\nu_m}(t)\right)}\right] = = \int_0^\infty e^{-c^2 s\|\xi\|^{2\beta}}\left[-\frac{\partial}{\partial s}\int_0^t h_{\nu_1,\,\dots,\,\nu_m}(z,s)\,dz\right] ds \tag{20}$$

We now take the Laplace transform of (20); we have that

$$
\begin{aligned}
&\int_0^\infty e^{-\mu t}\,\mathbb{E}\left[e^{i\vec{\xi}\cdot S_n^{2\beta}\left(c^2\mathcal{L}^{\nu_1,\,\dots,\,\nu_m}(t)\right)}\right]dt = \\
&= \int_0^\infty e^{-\mu t} dt \int_0^\infty e^{-c^2 s\|\xi\|^{2\beta}}\left[-\frac{\partial}{\partial s}\int_0^t h_{\nu_1,\dots,\nu_m}(z,s)\,dz\right] ds \\
&= \int_0^\infty e^{-c^2 s\|\xi\|^{2\beta}}\left[-\frac{\partial}{\partial s}\int_0^\infty h_{\nu_1,\dots,\nu_m}(z,s)\,dz\int_z^\infty e^{-\mu t}dt\right] ds \\
&= \int_0^\infty e^{-c^2 s\|\xi\|^{2\beta}}\left(-\frac{1}{\mu}\right)\frac{\partial}{\partial s}\left[\int_0^\infty e^{-\mu z} h_{\nu_1,\dots,\nu_m}(z,s)\,dz\right] ds \\
&= -\frac{1}{\mu}\int_0^\infty e^{-c^2 s\|\xi\|^{2\beta}}\frac{\partial}{\partial s}\mathbb{E}\left[e^{-\mu\sum_{j=1}^m \lambda_j^{\frac{1}{\nu_j}} H_j^{\nu_j}(s)}\right] ds
\end{aligned}
\tag{21}
$$

For the independence of the stable subordinators $H_j^{\nu_j}$ we have that

$$\mathbb{E}\left[e^{-\mu \sum_{j=1}^m \lambda_j^{\frac{1}{\nu_j}} H_j^{\nu_j}(s)}\right] = \prod_{j=1}^m \mathbb{E}\left[e^{-\mu \lambda_j^{\frac{1}{\nu_j}} H_j^{\nu_j}(s)}\right] = e^{-s \sum_{j=1}^m \lambda_j \mu^{\nu_j}} \quad (22)$$

and thus, by inserting (22) into (21) we get

$$\begin{aligned}
&\int_0^\infty e^{-\mu t}\, \mathbb{E}\left[e^{i\vec{\xi}\cdot S_n^{2\beta}\left(c^2 \mathcal{L}^{\nu_1,\dots,\nu_m}(t)\right)}\right] dt = \\
&= \frac{\sum_{j=1}^m \lambda_j \mu^{\nu_j}}{\mu} \int_0^\infty e^{-c^2 s \|\xi\|^{2\beta} - s \sum_{j=1}^m \lambda_j \mu^{\nu_j}} ds \\
&= \frac{\sum_{j=1}^m \lambda_j \mu^{\nu_j - 1}}{\sum_{j=1}^m \lambda_j \mu^{\nu_j} + c^2 \|\xi\|^{2\beta}}
\end{aligned}$$

which coincides with (16). This proves the statement of the theorem.

For $\beta = 1$ we have that

$$\begin{cases} \sum_{j=1}^{m} \lambda_j \frac{\partial^{\nu_j}}{\partial t^{\nu_j}} w(x_1, \dots, x_n; t) = c^2 \Delta w(x_1, \dots, x_n; t) \\ w(x_1, \dots, x_n; 0) = \delta(x_1, \dots, x_n) \end{cases} \tag{23}$$

and the fundamental solution coincides with the law of a subordinated n-dimensional Brownian motion

$$B_n \left(c^2 \mathcal{L}^{\nu_1, \dots, \nu_m}(t) \right) \tag{24}$$

For $m = 2$, $\nu_1 = 2\nu$, $\nu_2 = \nu$ we have the equation

$$\left(\frac{\partial^{2\nu}}{\partial t^{2\nu}} + 2\lambda \frac{\partial^{\nu}}{\partial t^{\nu}}\right) w(x_1, \dots, x_n; t) = -c^2(-\Delta)^{\beta} w(x_1, \dots, x_n; t) \tag{25}$$

which is the most immediate space-time extension of the classical telegraph equation.

In this case the fundamental solution of (23) coincides with the distribution of

$$W_n(t) = S_n^{2\beta}\left(c^2 \mathcal{L}^{\nu}(t)\right) \tag{26}$$

where

$$\mathcal{L}^{\nu}(t) = \inf\left(s \geq 0 : \mathcal{H}(s) = H_1^{2\nu}(s) + (2\lambda)^{\frac{1}{\nu}} H_2^{\nu}(s)\right)$$

for $0 < \nu < \frac{1}{2}$, $\beta \in (0, 1]$, where $H_1^{2\nu}$ and H_2^{ν} are independent subordinators.

The Fourier transform of $W_n(t)$ has the following expression

$$\mathbb{E}\left[e^{i\vec{\xi}\cdot W_n(t)}\right] = \frac{1}{2}\left[\left(1 + \frac{\lambda}{\sqrt{\lambda^2 - c^2\|\xi\|^{2\beta}}}\right) E_{\nu,1}(r_1 t^{\nu}) + \left(1 - \frac{\lambda}{\sqrt{\lambda^2 - c^2\|\xi\|^{2\beta}}}\right) E_{\nu,1}(r_2 t^{\nu})\right] \qquad (27)$$

where

$$r_1 = -\lambda + \sqrt{\lambda^2 - c^2\|\xi\|^{2\beta}}$$
$$r_2 = -\lambda - \sqrt{\lambda^2 - c^2\|\xi\|^{2\beta}}$$

and $E_{\nu,1}(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\nu k+1)}$ is the one-parameter Mittag-Leffler function.

The technique which permits to obtain (27) consists in the decomposition of the Fourier-Laplace transform as follows

$$\int_0^{\infty} e^{-\mu t} \int_{-\infty}^{+\infty} e^{i\vec{\xi}\cdot\vec{x}} u(\vec{x},t)\, d\vec{x}\, dt = \frac{\mu^{2\nu-1} + 2\lambda\mu^{\nu-1}}{\mu^{2\nu} + 2\lambda\mu^{\nu} + c^2\|\xi\|^{2\beta}} \tag{28}$$

$$= \frac{\mu^{\nu-1}}{\mu^{\nu} - r_1} + \frac{\mu^{\nu-1}}{\mu^{\nu} - r_2} - \left[\frac{\mu^{\nu-(1-\nu)}}{\mu^{\nu} - r_1} - \frac{\mu^{\nu-(1-\nu)}}{\mu^{\nu} - r_2}\right] \frac{1}{2\sqrt{\lambda^2 - c^2\|\xi\|^{2\beta}}}$$

and then consider that

$$\int_0^{\infty} e^{-\mu t} E_{\nu,1}(r_j t^{\nu})\, dt = \frac{\mu^{\nu-1}}{\mu^{\nu} - r_j} \tag{29}$$

$$\int_0^{\infty} e^{-\mu t} t^{(1-\nu)-1} E_{\nu,1-\nu}(r_j t^{\nu})\, dt = \frac{\mu^{2\nu-1}}{\mu^{\nu} - r_j}$$

Therefore, in view of (29), the inverse Laplace transform of (28) becomes

$$\int_{-\infty}^{+\infty} e^{i\vec{\xi}\cdot\vec{x}} u(\vec{x},t)\,d\vec{x} = E_{\nu,1}(r_1 t^\nu) + E_{\nu,1}(r_2 t^\nu) - \frac{t^{-\nu}}{2\sqrt{\lambda - c^2\|\xi\|^{2\beta}}} \times [E_{\nu,1-\nu}(r_1 t^\nu) - E_{\nu,1-\nu}(r_2 t^\nu)] \tag{30}$$

Since

$$E_{\nu,1-\nu}(z) = z\,E_{\nu,1}(z) + \frac{1}{\Gamma(1-\nu)}$$

with some further calculations we obtain (27).

Note that for $\beta = 1$, $\nu = 1$ the expression (27) coincides with the characteristic function of the symmetric telegraph process $T(t)$.

For $\beta = 1$, $\nu = \frac{1}{2}$ we have instead that (27) is the characteristic function of the time-changed telegraph process $T(|B(t)|)$, where $|B(t)|$ is a reflecting Brownian motion independent from T.
It is also true that for $\beta = 1$, $\nu = \frac{1}{2}$ the expression (27) is the characteristic function of the time-changed Brownian motion

$$W_1(t) = B\left(c^2 \mathcal{L}^{\frac{1}{2}}(t)\right) \tag{31}$$

where $\mathcal{L}^{\frac{1}{2}}(t)$ is the inverse of $\mathcal{H}^{\frac{1}{2}}(t) = t + (2\lambda)^2 H^{\frac{1}{2}}(t)$ and $H^{\frac{1}{2}}(t)$ is the stable subordinator of order $\frac{1}{2}$.
Thus we have the following equality in distribution

$$T(|B(t)|) \stackrel{i.d.}{=} B\left(c^2 \mathcal{L}^{\frac{1}{2}}(t)\right) \tag{32}$$

(see D'Ovidio et al. (2014)).

A similar relationship can be developed for the planar random motion evolving with velocity c, changing direction at Poisson-paced times and with uniformly distributed orientation of the deplacements.
This process $\mathcal{T}(t) = (X(t), Y(t))$ has distribution, at time t, concentrated in the circle $\{x, y : x^2 + y^2 \leq c^2 t^2\}$. The circumference ∂C_{ct} of radius ct is attained with probability $e^{-\lambda t}$ (distributed uniformly on ∂C_{ct}) and the inner points are reached with probability

$$r(x, y; t) = \frac{\lambda}{2\pi c} \frac{e^{-\lambda t + \frac{\lambda}{c}\sqrt{c^2 t^2 - x^2 - y^2}}}{\sqrt{c^2 t^2 - x^2 - y^2}}, \quad x^2 + y^2 < c^2 t^2,\ t > 0 \quad (33)$$

The function $r(x, y; t)$ satisfies

$$\left(\frac{\partial^2}{\partial t^2} + 2\lambda \frac{\partial}{\partial t} \right) r(x, y; t) = c^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) r(x, y; t) \qquad (34)$$

Instead the process

$$Q(t) = \mathcal{T}(|B(t)|)$$

satisfies the two-dimensional time-fractional equation

$$\left(\frac{\partial}{\partial t} + 2\lambda \frac{\partial^{\frac{1}{2}}}{\partial t^{\frac{1}{2}}} \right) q(x, y; t) = c^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) q(x, y; t) \qquad (35)$$

It is also true that $Q(t) = B_2 \left(c^2 \mathcal{L}^{\frac{1}{2}}(t) \right)$, where B_2 is the planar Brownian motion, and thus we have that

$$B_2 \left(c^2 \mathcal{L}^{\frac{1}{2}}(t) \right) \overset{i.d.}{=} \mathcal{T}(|B(t)|) \qquad (36)$$

In higher dimensions we are not able to provide results similar to the above ones because (with the exception of the space $\mathbb{R}^4$) we cannot give explicit and reasonable expressions for the distribution of n-dimensional random flights.

Iterated Brownian motions

The iterated Brownian motion

$$I(t) = B_1(|B_2(t)|) \tag{37}$$

has distribution solving the fractional equation

$$\frac{\partial^{\frac{1}{2}}}{\partial t^{\frac{1}{2}}} p(x,t) = \frac{1}{2^{\frac{3}{2}}} \frac{\partial^2}{\partial x^2} p(x,t) \tag{38}$$

The n-times iterated Brownian motion

$$I_n(t) = B_1\left(|B_2\left(\ldots |B_{n+1}(t)| \ldots\right)|\right) \tag{39}$$

has distribution $p_n(x,t)$ satisfying

$$\frac{\partial^{\frac{1}{2^n}}}{\partial t^{\frac{1}{2^n}}} p_n(x,t) = 2^{\frac{1}{2^n}-2} \frac{\partial^2}{\partial x^2} p_n(x,t) \tag{40}$$

The n-times iterated telegraph process

$$T_n(t) = T\left(\left|B_1\left(\left|B_2\left(\ldots\left|B_{n+1}(t)\right|\ldots\right)\right|\right)\right|\right) \tag{41}$$

has distribution $q_n(x,t)$ satisfying the equation

$$\frac{\partial^{\frac{2}{2^n}}}{\partial t^{\frac{2}{2^n}}} q_n(x,t) + 2\lambda \frac{\partial^{\frac{1}{2^n}}}{\partial t^{\frac{1}{2^n}}} q_n(x,t) = c^2 \frac{\partial^2}{\partial x^2} q_n(x,t) \tag{42}$$

For $n \to \infty$ the distribution of $I_n(t)$ becomes

$$\lim_{n\to\infty} p_n(x,t) = e^{-2|x|} \tag{43}$$

independent of t.

The distribution $q_n(x,t)$ converges as $n \to \infty$

$$\lim_{n\to\infty} q_n(x,t) = \frac{\sqrt{1+2\lambda}}{2c}\, e^{-\frac{|x|}{c}\sqrt{1+2\lambda}} \tag{44}$$

In the n-dimensional case we can reformulate the problem as follows. We take n-dimensional time-changed isotropic stable process (Brownian motion for $\beta = 1$) $S_n^{2\beta}\left(\mathcal{L}_r^{\nu_1,\ldots,\nu_m}(t)\right)$, where

$$\mathcal{L}_r^{\nu_1,\ldots,\nu_m}(t) = \inf\left(s : \mathcal{H}_r^{\nu_1,\ldots,\nu_m}(s) \geq t\right)$$

and

$$\mathcal{H}_r^{\nu_1,\ldots,\nu_m}(t) = \sum_{j=1}^{m} \lambda_j^{\frac{1}{\nu_j}} {}_1H^{\nu_j}\left({}_2H^{\nu_j}(\ldots {}_rH^{\nu_j}(t))\right)$$

is a combination of r-times iterated subordinators.
The law $w_{\nu_1,\ldots,\nu_m}(\vec{x}, t)$ of $S_n^{2\beta}\left(\mathcal{L}_r^{\nu_1,\ldots,\nu_m}(t)\right)$ satisfies

$$\begin{cases} \sum_{j=1}^{m} \lambda_j \dfrac{\partial^{\nu_j}}{\partial t^{\nu_j}} w_{\nu_1,\ldots,\nu_m}^{\beta,r}(\vec{x}, t) = -c^2(-\Delta)^{\beta} w_{\nu_1,\ldots,\nu_m}^{\beta,r}(\vec{x}, t) \\ w_{\nu_1,\ldots,\nu_m}^{\beta,r}(\vec{x}, 0) = \delta(\vec{x}) \end{cases}$$

For $r \to \infty$,

$$\lim_{r\to\infty} S_n^2\left(\mathcal{L}_r^{\nu_1,\ldots,\nu_m}(t)\right) = \lim_{r\to\infty} B_n\left(\mathcal{L}_r^{\nu_1,\ldots,\nu_m}(t)\right) \tag{45}$$

converges to the distribution

$$w_n(\vec{x}) = \frac{1}{(2\pi)^{\frac{n}{2}}}\left(\frac{\sqrt{\sum_{j=1}^m \lambda_j}}{c}\right)^{\frac{n+2}{2}} \|x\|^{-\frac{n-2}{2}} K_{\frac{n-2}{2}}\left(\frac{\sqrt{\sum_{j=1}^m \lambda_j}}{c}\|x\|\right) \tag{46}$$

where $K_\nu(z) = \int_0^\infty e^{-z\cosh t}\cosh \nu t\, dt$ is the modified Bessel function.

The function $w_n(\vec{x})$ satisfies

$$\left(\sum_{j=1}^{m}\lambda_j\right) w_n(x_1, \ldots, x_n) = c^2 \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2} w_n(x_1, \ldots, x_n) \tag{47}$$

For $n = 1$, formula (46) becomes

$$w_m(x) = \frac{\sqrt{\sum_{j=1}^{m}\lambda_j}}{2c}\, e^{-\frac{\sqrt{\sum_{j=1}^{m}\lambda_j}}{c}|x|} \tag{48}$$

which confirms the previous results.

References

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