About convergence rate of *extended* $M|G|\infty$ queueing system

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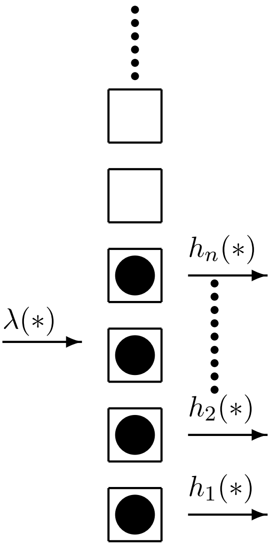

Full system state

The full system state at the time t includes the elapsed time $x_t^{(0)}$ from the last customer input, and elapsed times $x_t^{(i)}$ of the service of all customers staying in the system; for convenience, we also use n_t – the number of the customers in the system at the time t.
So, the full system state is described by the vector $X_t = \left(n_t, x_t^{(0)}; x_t^{(1)}, x_t^{(2)}, \ldots, x_t^{(n_t)}\right)$ – for simplicity, $x_t^{(i)} \geq x_t^{(i+1)}$ for $i = 1, \ldots, n_t$.
It is easy, that $x_t^{(0)} \leqslant x_t^{(n_t)}$, i.e. the customer numbered by n_t appeared in the system no later than at the time $t - x_t^{(0)}$.
So, we suppose that the intensity of input flow is $\lambda = \lambda(X_t)$, and intensity of service for i-th customer is $h_i = h_i(X_t)$.

$$X_0 = (0; 0).$$

Conditions

- For all possible X_t, $0 < \lambda_0 \leqslant \lambda(X_t) \leqslant \Lambda < \infty$;
- For all possible $X_t = \left(n_t, x_t^{(0)}; x_t^{(1)}, x_t^{(2)}, \ldots, x_t^{(n_t)}\right)$, $h_i(X_t) \geqslant \dfrac{C}{1 + x_t^{(i)}}$ — for some $C > 2$.

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About intensities

If some object lives a random time τ with the distribution function $G(s)$, and it is alive at the time t, then the probability of its dead in the interval $(t, t+\Delta)$ is equal

$$\frac{G(t+\Delta) - G(t)}{1 - G(t)} = \frac{G'(t)}{1 - G(t)}\Delta + o(\Delta); \qquad h(t) \stackrel{\text{def}}{=\!=} \frac{G'(t)}{1 - G(t)};$$

and $G(t) = 1 - \exp\left(-\int\limits_0^t h(s)\,\mathrm{d}s\right)$.

The process X_t is Markov.

For small time $\Delta > 0$ and given X_t,

$$\mathbf{P}\left\{ \begin{array}{l} n_{t+\Delta} = n_t + 1,\ x^{(0)}_{t+\Delta} = x^{(n_{t+\Delta})}_{t+\Delta} \in (0;\Delta), \\ \text{and } x^{(i)}_{t+\Delta} = x^{(i)}_t + \Delta \text{ for all } i = 1, \ldots, n_t \end{array} \right\} = \lambda(X_t)\Delta + o(\Delta);$$

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$$\mathbf{P}\{n_{t+\Delta} = n_t - 1\} = \sum_{i=1}^{n_t} h_i(X_t)\Delta + o(\Delta); \quad \text{and for all } i = 1, 2, \dots, n_t$$

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$$\mathbf{P}\left\{ n_{t+\Delta} = n_t; x^{(i)}_{t+\Delta} = x^{(i)}_t + \Delta \text{ for all } i = 0, \dots, n_t \right\} = 1 - \left(\lambda(X_t) + \sum_{i=1}^{n_t} h_i(X_t) \right) \Delta + o(\Delta).$$

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This conditions guarantee that X_t is a Markov process on the state space

$$\mathscr{X} \stackrel{\text{def}}{=} \bigcup_{n=1}^{\infty} \mathscr{S}_i, \text{ where } \mathscr{S}_i \stackrel{\text{def}}{=} \mathbb{Z}_+ \times \prod_{j=1}^{i} \mathbb{R}_+.$$

The process X_t is regenerative

Moreover, X_t is regenerative process: the regeneration points are the times where $X_t = (1;0;0)$, i.e. before this time the system was free, and at this time the customer comes into idle system.

Denote R_1, R_2, R_3, ... the length of sequential regeneration periods. These random variables are i.i.d., they consist of the idle period σ_i and busy period ζ_i; $\mathsf{E}\,\sigma_i^k \leqslant \dfrac{k!}{\lambda_0^k}$.

Fact

It is well-known, that if $\mathsf{E}\,(\sigma_i+\zeta_i)^k<\infty$ for some $k>1$, then the rate of convergence of the distribution of the regenerative process – to the stationary distribution – is less that $\dfrac{\mathbb{K}}{t^{k-1}}$ for *some* $\mathbb{K}$.

If we don't know the constant $\mathbb{K}$, we can not use this fact in the practice.

Our goal is the upper bounds for this constant $\mathbb{K}$.

Consider the "standard" queueing system $M|G|\infty$ where the input flow has the constant intensity Λ, random service times have the same distribution function $G_0(s) = 1 - \dfrac{1}{(1+s)^C}$ – in this case $h_0(s) = \dfrac{C}{1+s}$.
The times between the arrivals of customers and service times are mutually independent.
Like the previous, for this system we construct the Markov process $Y_t = \left(m_t; y_t^{(0)}; y_t^{(1)}, y_t^{(2)}, \ldots, y_t^{(n_t)}\right)$, where m_t is the number of the customers in the system at the time t, $y_t^{(0)}$ is the elapsed time from the last customer input, and $y_t^{(i)}$ is an elapsed times of the service of i-th customer being in the system; $Y_0 = (0; 0)$.

Lemma

It is possible to create the processes X_t and Y_t on the same probability space by such a way that:

1. $X_0 = Y_0 = (0;0)$;
2. *for all* $t \geq 0$, $n_t \leqslant m_t$;
3. *for all* $t \geq 0$ *there are the numbers* $k_1 < k_2 < \cdots < k_{n_t} \leqslant m_t$ *such that* $x_t^{(i)} \leqslant y_t^{(k_i)}$.

Definition

We say that $\xi \prec \eta$ if $\mathbf{P}\{\xi < s\} \geqslant \mathbf{P}\{\eta < s\}$ for all $s \in \mathbb{R}$.

So, the busy period ζ^X of X_t is less then busy period ζ^Y of Y_t in the sense of this definition: $\zeta^X \prec \zeta^Y$; moreover, $\mathbf{P}\{n_t = 0\} \geqslant \mathbf{P}\{m_t = 0\}$, and $\mathsf{E}\,[\zeta^X]^k \leqslant \mathsf{E}\,[\zeta^Y]^k$ for all $k > 0$.

What we know about “standard” system $M|G|\infty$

In “standard” system $M|G|\infty$, the intensity of input flow is constant Λ, the service times ξ_i are i.i.d.r.v. with distribution function $\mathbf{P}\{\xi_i < s\} = G_0(s) = 1 - \dfrac{1}{(1+s)^C}$.

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- $\mathbf{P}\{m_t = k\} = \dfrac{e^{-\mathfrak{G}(t)}\left(\mathfrak{G}(t)\right)^k}{k!}$, where $\mathfrak{G}(t) \stackrel{\text{def}}{=\!=} \Lambda \int\limits_0^t (1 - G_0(s))\,\mathrm{d}s$.

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  So, for all $t \geqslant 0$, $\mathbf{P}\{m_t = 0\} \geqslant \lim\limits_{t\to\infty} \mathbf{P}\{m_t = 0\} = e^{-\rho}$, where

  $\rho \stackrel{\text{def}}{=\!=} \Lambda m_1 = \Lambda \int\limits_0^\infty (1 - G_0(s))\,\mathrm{d}s$.

What we know about "standard" system $M|G|\infty$

- For the busy period of the considering system $M|G|\infty$ (denote the busy periods by ζ_i) we know the distribution function

$$B(x) \stackrel{\text{def}}{=} \mathbf{P}\{\zeta_i \leqslant x\} = 1 - \frac{1}{\Lambda} \sum_{k=1}^{\infty} c^{n*}(x),$$

where $c(x) \stackrel{\text{def}}{=} \Lambda(1 - G_0(x))e^{-\mathfrak{G}(x)}$, and c^{n*} is a n-th convolution of the function $c(x)$.

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- Moreover, we know the Laplace transform of $B(x)$:

$$\mathcal{L}[B](s) = 1 + \frac{s}{\Lambda} - \frac{1}{\Lambda \int\limits_0^\infty e^{-st - \Lambda \int\limits_0^t [1 - G_0(v)]\,\mathrm{d}v}\,\mathrm{d}t}.$$

- Also we have formulae:

$$\mathsf{E}\,\zeta_i^n = (-1)^{n+1}\left\{\frac{e^\rho}{\Lambda} n C^{(n-1)} - e^\rho \sum_{k=1}^{n-1} C_n^k \mathsf{E}\,\zeta_i^{n-k} C^{(p)}\right\}, \qquad n \in \mathbb{N}, \text{ where}$$

$$C^{(n)} = \int_0^\infty (-t)^n \left(e^{-\Lambda \int_0^\infty [1-G_0(v)]\,\mathrm{d}v} \right) \Lambda[1-G_0(t)]\,\mathrm{d}t.$$

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Thus, we have $\mathsf{E}\,\zeta_i = \dfrac{e^\rho - 1}{\Lambda}$ and $\mathsf{E}\,\zeta_i^2 = \dfrac{2e^{2\rho}}{\Lambda} \displaystyle\int_0^\infty \left(e^{-\Lambda \int_0^\infty [1-G_0(v)]\,\mathrm{d}v} \right) \mathrm{d}t.$

But for the next moments of ζ_i the calculations are very complicated, and we want to *estimate* this moments.

Estimation

Return to the formula $B(x) \stackrel{\text{def}}{=\!=} \mathbf{P}\{\zeta_i \leqslant x\} = 1 - \dfrac{1}{\Lambda}\sum\limits_{k=1}^{\infty} c^{n*}(x)$, where

$c(x) \stackrel{\text{def}}{=\!=} \Lambda(1 - G_0(x))e^{-\mathfrak{G}(x)}$, and c^{n*} is a n-th convolution of the function $c(x)$.

It is easy, $\displaystyle\int\limits_0^{\infty} c(s)\,\mathrm{d}s = 1 - e^{-\Lambda m_1} = 1 - e^{-\rho} \stackrel{\text{def}}{=\!=} \varrho \in (0;1)$.

So, $r(s) \stackrel{\text{def}}{=\!=} \dfrac{c(s)}{\varrho}$ is a density of distribution of some r.v. ς. And $c^{n*}(x) = \varrho^n r_n(x)$

where $r_n(x)$ is a density of distribution of the sum $\sum\limits_{i=1}^{n} \varsigma_i$ where ς_i are i.i.d.r.v. with

distribution density $r(s)$.

Further, $\mathsf{E}\,\varsigma^k = \displaystyle\int\limits_0^{\infty} s^k \varrho^{-1}\Lambda(1 - G_0(s))e^{-\mathfrak{G}(s)}\,\mathrm{d}s \leqslant \frac{\Lambda}{\varrho}\int\limits_0^{\infty} s^k(1 - G_0(s))\,\mathrm{d}s = \frac{\Lambda \mathsf{E}\,\xi_i^{k+1}}{(k+1)\varrho}$.

Estimation

Now, using the Jensen's inequality in the form $(a_1+\ldots+a_n)^k \leqslant n^{k-1}(a_1^k+\ldots+a_n^k)$, we have:

$$\mathsf{E}\,\zeta_i^k = \int\limits_0^\infty kx^{k-1}(1-B(x))\,\mathrm{d}x = \frac{1}{\Lambda}\sum_{n=1}^\infty \int\limits_0^\infty kx^{k-1}c^{n*}(x)\,\mathrm{d}x =$$

$$= \frac{1}{\Lambda}\sum_{n=1}^\infty \int\limits_0^\infty kx^{k-1}\varrho^n r^{n*}(x)\,\mathrm{d}x = \frac{1}{\Lambda}\sum_{n=1}^\infty \varrho^n k\,\mathsf{E}\left(\sum_{j=1}^n \varsigma_j\right)^{k-1} \leqslant$$

$$\leqslant \frac{1}{\Lambda}\sum_{n=1}^\infty \varrho^n k n^{k-1}\frac{\Lambda \mathsf{E}\,\xi_i^k}{\varrho k} = \frac{\mathsf{E}\,\xi_i^k}{\varrho}\sum_{n=1}^\infty n^{k-1}\varrho^n = \frac{\mathsf{E}\,\xi_i^k}{\varrho}\times\varphi(\varrho,k-1),$$

where $\varphi(x,k) \stackrel{\mathrm{def}}{=\!=} \left(x\dfrac{\mathrm{d}}{\mathrm{d}x}\right)^k \dfrac{1}{1-x}$.

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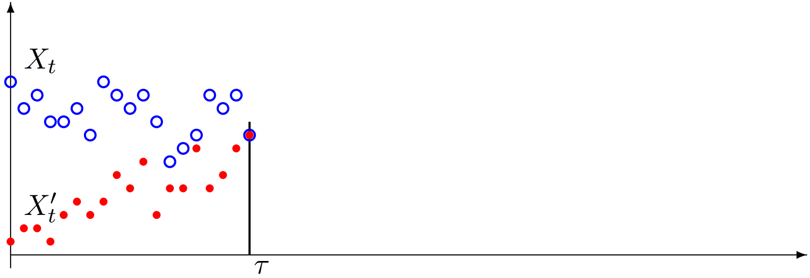

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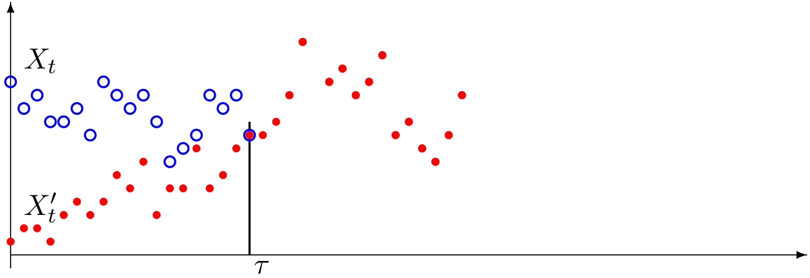

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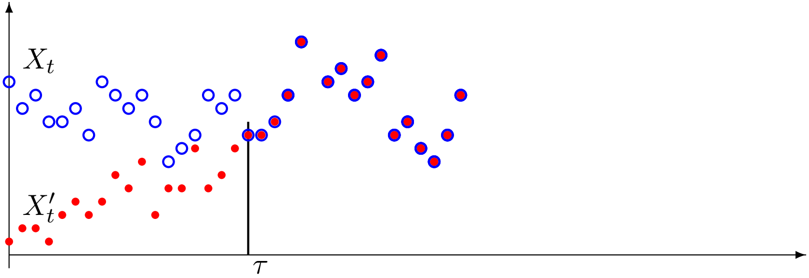

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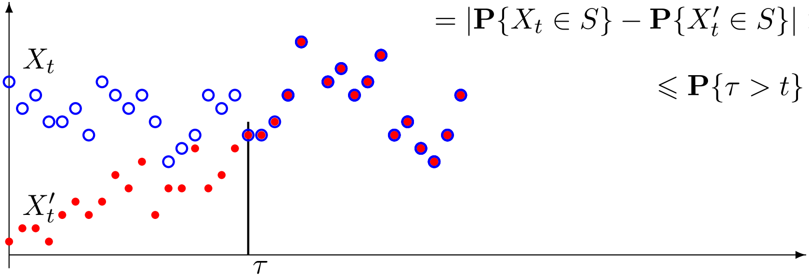

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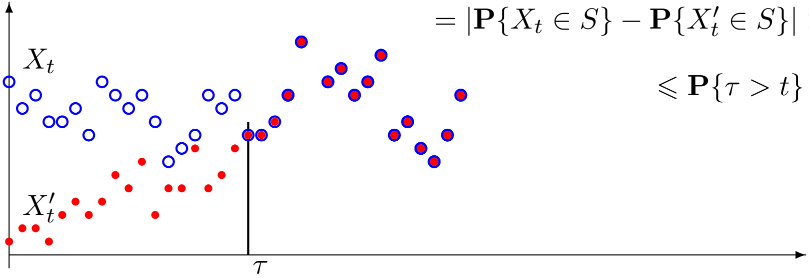

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For the processes in continuous time "direct" coupling method is impossible.

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(ii) For all $t > \tau(X_0, X'_0) = \tau(Z_0, Z'_0) \stackrel{\text{def}}{=\!=} \inf\{t \geqslant 0 :\ Z_t = Z'_t\}$ the equality $Z_t = Z'_t$ is true.

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(iii) For all $X_0, X'_0 \in \mathscr{X}$, $\mathbf{P}\{\tau(X_0, X'_0) < \infty\} = 1$.

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(iii) For all $X_0, X'_0 \in \mathscr{X}$, $\mathbf{P}\{\tau(X_0, X'_0) < \infty\} = 1$.

If the conditions (i)–(iii) are satisfied, then the paired process $\mathcal{Z}_t$ is called *successful coupling*.

$$\text{Hence,} \quad |\mathbf{P}\{X_t \in S\} - \mathbf{P}\{X'_t \in S\}| = |\mathbf{P}\{Z_t \in S\} - \mathbf{P}\{Z'_t \in S\}| \leqslant$$

$$\leqslant \mathbf{P}\{\tau > t\} \leqslant \frac{\mathsf{E}\,\phi(\tau)}{\phi(\tau)} = \mathscr{R}(t, X_0, X'_0), \qquad \phi\uparrow, \quad \phi > 0.$$

And $\|\mathscr{P}_t - \mathscr{P}'_t\|_{TV} \stackrel{\text{def}}{=\!=} 2 \sup\limits_{S \in \mathscr{B}(\mathscr{X})} |\mathbf{P}\{X_t \in S\} - \mathbf{P}\{X'_t \in S\}| \leqslant 2\mathscr{R}(t, X_0, X'_0)$.

Let $X_0=(0;0)$, and X'_0 is arbitrary from $\mathscr{X}$;
$X_t=\left(n_t,x_t^{(0)};x_t^{(1)},x_t^{(2)},\ldots,x_t^{(n_t)}\right)$;
$X'_t=\left(n'_t,x_t^{(0)'};x_t^{(1)'},x_t^{(2)'},\ldots,x_t^{(n'_t)'}\right)$.
Consider the times θ_1, θ_2, $\theta_3,\ldots$ when X'_t comes to the set $\mathscr{S}_1$, or n'_t changes the value from 1 to 0.

$$\mathbf{P}\{X_{\theta_i}\in\mathscr{S}_1\}=\mathbf{P}\{n_{\theta_i}=0\}\geqslant\mathbf{P}\{m_{\theta_i}=0\}\geqslant e^{-\rho}\stackrel{\text{def}}{=\!=}\pi_0.$$

If $n_{\theta_i+}=n'_{\theta_i+}$ then both systems are idle, and incoming flow has the common part with constant intensity $\lambda_0>0$. The not-common parts of these flows are in $(0;\Lambda-\lambda_0)$.
Therefore one can use the Coupling Lemma for construct the prolongation of X_t and X'_t by such a way that the following comings of the customers to the both system are simultaneous – with probability greater than $\pi_1\stackrel{\text{def}}{=\!=}\dfrac{\lambda_0}{\Lambda}$.
If this event happens, the processes will stick together.

Coupling Lemma

Lemma (Coupling Lemma)

Let $f_i(s)$ *be the distribution density of r.v.* θ_i *(*$i=1,2$*). And let*
$$\int\limits_{-\infty}^{\infty} \min(f_1(s), f_2(s))\,\mathrm{d}s = \varkappa > 0.$$
Then on some probability space there exists two random variables ϑ_i *such that* $\vartheta_i \stackrel{\mathscr{D}}{=} \theta_i$*, and* $\mathbf{P}\{\vartheta_1 = \vartheta_2\} \geqslant \varkappa$*.*

Constructive proof.

Let $\mathscr{U}_i$ be independent uniform r.v. on $(0;1)$; $\phi(s) \stackrel{\mathrm{def}}{=} \varkappa^{-1} \int\limits_{-\infty}^{s} \min(f_1(s), f_2(s))\,\mathrm{d}s$,

$\psi_i \stackrel{\mathrm{def}}{=} (1-\varkappa)^{-1} \left(\int\limits_{-\infty}^{s} f_i(s) - \min(f_1(s), f_2(s))\,\mathrm{d}s \right)$, then

$\vartheta_i \stackrel{\mathrm{def}}{=} \mathbf{1}(\mathscr{U}_0 < \varkappa)\phi^{-1}(\mathscr{U}_4) + \mathbf{1}(\mathscr{U}_0 \geqslant \varkappa)\psi_i^{-1}(\mathscr{U}_i)$. □

Coupling epoch

So, at the end of any regeneration period, the processes X_t and X'_t can coincides with probability $\pi \stackrel{\text{def}}{=\!=} \pi_0\pi_1$ (this probability can be improved). Therefore the coupling epoch $\tau(X'_0)$ is a geometrical sum of the regeneration periods of X'_t including the first incomplete one.

Thus, we can find the upper bounds for $\mathsf{E}\,[\tau(X'_0)]^k$ in the case when $\mathsf{E}\,\xi_i^k<\infty$ $(C>k-1)$:

$$\mathsf{E}\,[\tau(X'_0)]^k \leqslant \pi\sum_{i=0}^{\infty}(1-\pi)^i\mathsf{E}\left(R_0+\sum_{j=1}^{i}b_j\right)^k \leqslant \sum_{i=0}^{\infty}(1-\pi)^i(i+1)^{k-1}\mathsf{E}\left(R_0^k+\sum_{j=1}^{i}R_j^k\right) =$$

$$= \mathsf{E}\,R_0^k\times\sum_{i=0}^{\infty}(1-\pi)^i(i+1)^{k-1}+\mathsf{E}\,R_1^k\times\sum_{i=0}^{\infty}(1-\pi)^i(i+1)^k = K_0(k,\pi)\mathsf{E}\,R_0^k+K(k,\pi)\mathsf{E}\,R_1^k,$$

where R_0 is the first regeneration point of X'_t, and R_i, $i\in\mathbb{N}$ are the length of subsequent regeneration periods.

TV-distance

So, if $\mathscr{P}_t$ is distribution of X_t, and $\mathscr{P}'_t$ is distribution of X'_t then

$$\|\mathscr{P}_t - \mathscr{P}'_t\|_{TV} \leqslant 2\frac{K_0(k,\pi)\mathsf{E}\,R_0^k + K(k,\pi)\mathsf{E}\,R_1^k}{t^k}.$$

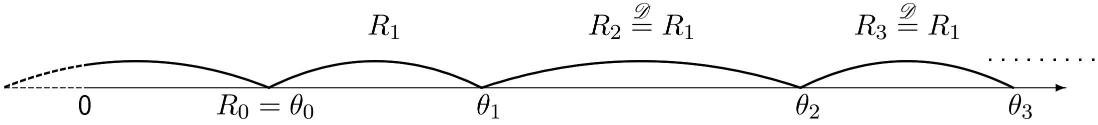
If $\mathscr{P}'_t \Longrightarrow \mathscr{P}$ for all X'_0, then

$$\|\mathscr{P}_t - \mathscr{P}\|_{TV} \leqslant 2\frac{\int\limits_{\mathscr{X}}(K_0(k,\pi)\mathsf{E}\,R_0^k + K(k,\pi)\mathsf{E}\,R_1^k)\mathscr{P}(\,\mathrm{d}X'_0)}{t^k} =$$

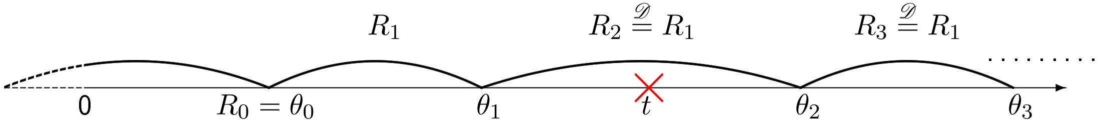
$$= 2\frac{K_0(k,\pi)\int\limits_{\mathscr{X}}\mathsf{E}\,R_0^k\mathscr{P}(\,\mathrm{d}X'_0) + K(k,\pi)\mathsf{E}\,R_1^k}{t^k}.$$

But we don't know neither the stationary distribution of X_t nor the stationary distribution for the "standard" model!!!

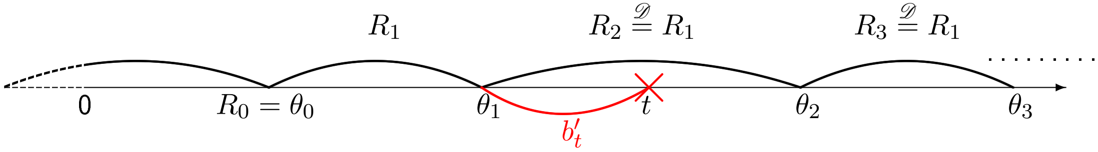
Embedded renewal process



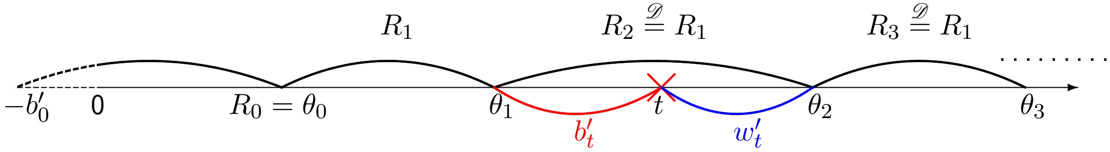
Embedded renewal process



Embedded renewal process



Embedded renewal process



There exists some map $\mathscr{Q} : \mathscr{B}(\mathbb{R}_+) \to \mathscr{B}(\mathscr{X})$ such that

$$b'_t \in A \subseteq \mathscr{B}(\mathbb{R}_+) \Rightarrow X'_t \in \mathscr{Q}(A) \subseteq \mathscr{B}(\mathscr{X}), \text{and vice versa.}$$

So, it is enough estimate the TV-distance between b'_t (delay backward renewal time) and b_t (non-delay backward renewal time).

If the regeneration period has the distribution function $\Phi(s) = \mathbf{P}\{R_1 < s\}$, then the stationary backward renewal time $\tilde{b}_t$ and forward renewal time $\tilde{w}_t$ have a same distribution function

$$\tilde{\Phi}(s) \overset{\text{def}}{=\!=} \frac{\int\limits_0^s (1 - \Phi(u))\, \mathrm{d}u}{\mathsf{E}\, R_1}.$$

TV-distance

Thus, returning to $\int\limits_{\mathscr{X}} \mathsf{E}\, R_0^k \mathscr{P}(\,\mathrm{d}X_0')$, we replace it by

$$\int\limits_{\mathscr{X}} \mathsf{E}\, R_0^k \mathscr{P}(\,\mathrm{d}X_0') = \int\limits_0^\infty R^k \,\mathrm{d}\tilde{\Phi}(R) = \frac{1}{\mathsf{E}\, R_1} \int\limits_0^\infty R^k (1 - \Phi(R))\,\mathrm{d}R = \frac{\mathsf{E}\, R_1^{k+1}}{(k+1)\mathsf{E}\, R_1}.$$

Finally, we can calculate the constant $\mathbb{K}(k)$ such that

$$\|\mathscr{P}_t - \mathscr{P}\|_{TV} \leqslant \frac{\mathbb{K}(k)}{t^k}$$

for $k < C - 2$ in conditions

- For all possible X_t, $0 < \lambda_0 \leqslant \lambda(X_t) \leqslant \Lambda < \infty$;
- For all possible $X_t = \left(n_t, x_t^{(0)}; x_t^{(1)}, x_t^{(2)}, \ldots, x_t^{(n_t)}\right)$, $h_i(X_t) \geqslant \dfrac{C}{1 + x_t^{(i)}}$ — for some $C > 2$.

ОБРАЩЕНИЕ К РУССКОЯЗЫЧНЫМ РОССИЯНАМ

Если в стране нет широкодоступной научной литературы, то наука в этой стране чахнет.

Без национальной терминологии национальная наука умирает.

Пока в России не было русской математической терминологии и русских книг по математике, математикой здесь занимались приглашённые иностранцы и (немного) их ученики. Без возникшего в начале XIX-го века движения российских интеллектуалов, сначала переводивших иностранные математические книги на русский язык, а потом и создававших оригинальные русские учебники и русские математические издания, – так вот, без этого не было бы у нас множества людей, изучающих математику, – и не было бы ни Остроградского, ни Чебышёва, ни Ковалевской.

Сейчас жизнь заставляет россиян публиковаться в западных журналах. Но, *ПОЖАЛУЙСТА*, дублируйте эти публикации на русском языке. Ведь часто мы уже не можем подобрать русский термин для знакомого по зарубежным публикациям понятия.

Ещё немного – и математика останется только “там”...

Thanks for your attention!