Weighted entropy and optimal portfolios for cautious Kelly investments

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1 Maximum entropy principle

Suppose you would like to estimate probabilities $\mathbf{p} = (p_1, p_2, \ldots, p_6)$ for a die based on the mean score $y = \mathbb{E}[X]$. MEP suggests to take the distribution with maximal entropy consistent with the available information. Let $N = (1, 2, 3, 4, 5, 6)^T$. Define the Lagrangian

$$\mathcal{L} = -\mathbf{p}^T \log \mathbf{p} + \lambda(y - N^T \mathbf{p}) + \mu(1 - \mathbf{p}^T \mathbf{1}). \tag{1.1}$$

This analysis implies

$$\hat{p}_j = \frac{e^{-j\hat{\lambda}(y)}}{Z(\hat{\lambda}(y))}, j = 1, 2, \ldots, 6, Z(\hat{\lambda}) = \sum_{k=1}^{6} e^{-k\hat{\lambda}}. \tag{1.2}$$

For $y = 3.5$ we have $\hat{\lambda}(3.5) = 0$ and distribution is uniform. For $y > 3.5$ we have $\hat{\lambda}(y) < 0$ and probabilities increase with the increase of the face number.

Many applied papers argue for that the severity of different factors (as well as probability) should be taken into account. This leads to

Weighted Maximum entropy principle: maximize $-\sum\limits_{k=1}^{K} w_k p_k \log p_k$. with restrictions.

Suppose you would like to estimate the probability distribution of different attacks on computer system based on the mean damage and some additional information. Applications of the WE and WDE to the security quantification of information systems are discussed in:

Paksakis C., Mermigas S., Pirourias S., Chondrokoukis G., The role of weighted entropy in security quantification, Int. Journ. of Information and Electronics Engineering (2013), **3**, 2, 156-159

Other domains range from the stock market to the image processing. We concentrate on the so-called Kelly investments. In real financial market the investment strategy depends not only on the current values of stocks but also on their dynamics. For simplicity, we start with the static market.

2 Numerical example

Initial capital \$1000. Five risky assets.
Toss a coin 700 times, 1000 realizations:

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P(head)	Odds	**P(choose)**	Kelly bets
0.57	1-1	0.1	0.14
0.38	1-2	0.3	0.07
0.285	1-3	0.3	0.047
0.228	1-4	0.2	0.035
0.19	1-5	0.1	0.028

(2.1)

...	Min	Max	Mean	Median
Kelly	18	483,883	48.135	17.269
1/2 Kelly	145	111,770	13.019	8.013

(2.2)

final	> 500	> 1000	> 5000	$> 50,000$	$> 100,000$
Kelly	916	870	598	302	166
1/2 Kelly	990	969	480	30	1

(2.3)

3 The cautious Kelly investments with a single risky asset

An investor is betting on results ε_n of subsequent random trials, $n = 0, 1, 2, \ldots$. Suppose that the ε_n are generated by a Markov chain with a finite or countable state space M. The transition matrix is $\mathtt{P} = (\mathtt{p}(i,k),\ i,k \in M)$. The recursion for Z_n is

$$Z_n = Z_{n-1} + C_{n-1}g(\varepsilon_{n-1}, \varepsilon_n) = Z_{n-1}\left[1 + \frac{C_{n-1}g(\varepsilon_{n-1}, \varepsilon_n)}{Z_{n-1}}\right]; \tag{3.1}$$

it shows that $Z_n \in \mathfrak{W}_n$: $Z_n = Z_n(\boldsymbol{\varepsilon}_0^n)$. For the return function g we will use the acronym RF.

Next, let us consider another function, $(i,k) \mapsto \varphi(i,k) \geq 0$, representing a 'utility' value assigned to outcome k when it succeeds outcome i. If $\varphi(i,k) \equiv 1$, all outcomes are treated entirely in terms of their returns, and if $\varphi(i,k)$ does not depend on i, the value does not take into account the history. We say that φ is a *weight function* (WF); including one-step history i agrees with the Markovian assumption for ε_n.

We wish to maximize, in $C_0, \ldots, C_{n-1}$, the mean value $\mathbb{E}S_n$ where

$$S_n := \sum_{j=1}^{n} \varphi(\varepsilon_{j-1}, \varepsilon_j) \ln \frac{Z_j}{Z_{j-1}}, \tag{3.2}$$

and determine, when possible, a sequence of optimal startegies $\{C_j^{\mathrm{O}}\}$, within a 'natural' classes $\{\mathcal{C}_j\}$ of predictable strategies defined by recursive inequalities (3.6) below:

$$(C_0^{\mathrm{O}}, \ldots, C_{n-1}^{\mathrm{O}}) = \operatorname{argmax}\left[\mathbb{E}S_n :\ C_j \in \mathcal{C}_j,\ 0 \leq j \leq n-1\right]. \tag{3.3}$$

The classes $\mathcal{C}_j$ are described through conditions (a0)–(a2) or (a0)–(a3) listed in Eqn (3.6) below. Under our assumptions, the optimum is at a proportional betting, where $C_{j-1}^{\mathrm{O}} = D_{j-1}^{\mathrm{O}}(\varepsilon_{j-1})Z_{j-1}$. Here Z_{j-1} is the capital after $j-1$ trials and $D_{j-1}^{\mathrm{O}}(i)$ is the proportionality coefficient indicating the fraction of the capital to be invested into the jth trial.

Quantity S_n/n can be considered as a weighted log-capital rate after n trials. When $\varphi(i,k) \equiv 1$, the sum in (3.2) becomes telescopic, and S_n equals $\ln \dfrac{Z_n}{Z_0}$ (a standard quantity in financial calculations, particularly in relation to the Kelly-type investments).

The form of summation in Eqn (3.2) suggests the use of the weighted Kullback–Leibler (KL) entropy of the row probability vector $\big(\mathtt{p}(i,k),\ k\in M\big)$ relative to chosen 'calibrating' functions $(i,k)\in M\times M\mapsto \mathtt{q}_j(i,k)>0$, $j=0,1,\ldots$. To this end, set:

$$\alpha_j(i)=\sum_{l\in M}\varphi(i,l)\mathtt{p}(i,l)\ln\frac{\mathtt{p}(i,l)}{\mathtt{q}_j(i,l)},\ i\in M,\ j\geq 0. \tag{3.4}$$

The choice of calibrating functions (CFs) $\mathtt{q}_j(i,k)$ is a part of the optimization procedure and is discussed below: see (3.6) and (3.7). We consider the random process (RP) of the cumulative weighted KL entropy

$$\mathrm{A}_n=\sum_{j=1}^n\alpha_{j-1}(\epsilon_{j-1}),\ \text{ with }\ \mathbb{E}\mathrm{A}_n=\sum_{j=1}^n\mathbb{E}\alpha_{j-1}(\varepsilon_{j-1}). \tag{3.5}$$

Also fix a value $b>0$ (a proportional ruin threshold).

Let us summarize conditions on the class of policies and involved functions: $\forall\ \ j\geq 0$,

$$\begin{array}{l}\text{(a0) } C_j\in\mathfrak{W}_j,\ \text{ (a1) } 0\leq C_j<Z_j,\ \text{ (a2) } 1+\dfrac{C_jg(\varepsilon_j,\varepsilon_{j+1})}{Z_j}\geq b,\ \text{ and}\\ \text{(a3)}\quad C_j(\boldsymbol{\varepsilon}_0^j)\sum\limits_{l\in M}\varphi(\varepsilon_j,l)\mathtt{q}_{j+1}(\varepsilon_j,l)g(\varepsilon_j,l)=0.\end{array} \tag{3.6}$$

Also, $\forall\ \ i\in M,\ j\geq 0$, we assume the condition labelled as (q–p) in Eqn (3.7):

$$(\mathrm{q}-\mathrm{p})\qquad \sum_{l\in M}\varphi(i,l)\big[\mathtt{q}_j(i,l)-\mathtt{p}(i,l)\big]\leq 0. \tag{3.7}$$

Theorem 1.1. *Suppose the recursion* (3.1) *holds true.*

(a) *Suppose a sequence of CFs* $\mathtt{q}_n$ *is given, obeying* (3.7). *Take any sequence* $\{C_n,\ n\geq 0\}$ *of random variables (RVs)* C_n *satisfying properties* (a0)–(a3) *in Eqn* (3.6). *Consider RVs* S_n *and* A_n *defined in* (3.2) *and* (3.4)–(3.5). *Then the sequence of differences* $\{S_n-\mathrm{A}_n,\ n\geq 1\}$ *is a supermartingale; consequently,* $\mathbb{E}\,S_n\leq\mathbb{E}\mathrm{A}_n\ \ \forall\ n\geq 1$.

(b) *To achieve equality* $\mathbb{E}\,S_n=\mathbb{E}\mathrm{A}_n$*: the sequence* $\{S_n-\mathrm{A}_n\}$ *is a martingale for a sequence of RVs* C_n *satisfying* (a0)–(a3) *in* (3.6) *iff the following conditions* (i)–(ii) *hold.*

(i) *There exists a function* $D:\ M\to\mathbb{R}$ *such that,* $\forall\ \ i,k\in M$,

$$\begin{array}{l}\text{(i1) } 0\leq D(i)<1,\ \text{ (i2) } 1+D(i)g(i,k)\geq b,\ \text{ (i3) } D(i)\sum\limits_{l\in M}\dfrac{\mathtt{p}(i,l)\varphi(i,l)g(i,l)}{1+D(i)g(i,l)}=0,\\ \qquad \textit{i.e., either } \text{(i3A)} \sum\limits_{l\in M}\dfrac{\mathtt{p}(i,l)\varphi(i,l)g(i,l)}{1+D(i)g(i,l)}=0\ \textit{ or } \text{(i3B) } D(i)=0,\ \textit{ and}\\ \text{(i4)}\quad \textit{the CFs } \mathtt{q}_j \textit{ are of the form } \mathtt{q}_j(i,k)=\dfrac{\mathtt{p}(i,k)}{1+D(i)g(i,k)},\ \ j=0,1,\ldots.\end{array} \tag{3.8}$$

(ii) $\forall\ \ n\geq 1$, *the policy* C_{n-1} *produces a proportional investment:* $C_{n-1}(\boldsymbol{\varepsilon}_0^{n-1})=D(\varepsilon_{n-1})Z_{n-1}$.

Furthermore, the CF values $\mathtt{q}_n(i,k)$ *given in* (i4) *satisfy* $\sum\limits_{l\in M}\mathtt{q}_n(i,l)=1$ *(which yields a transition probability matrices) iff, in addition to* (i3), *we have that*

$$D(i)=0\quad\text{or}\quad\sum_{l\in M}\frac{\mathtt{p}(i,l)g(i,l)}{1+D(i)g(i,l)}=0,\ \ i\in M. \tag{3.9}$$

(c) *Define the map* $i\in M\mapsto D^{\mathrm{O}}(i)$ *as follows. Given* i, *consider Eqn* (i3A)*: it has at most one solution* $D(i)>0$. *If* (i3A) *has a solution* $D(i)>0$ *obeying conditions* (i1)–(i2), *set* $D^{\mathrm{O}}(i)=D(i)$; *otherwise* $D^{\mathrm{O}}(i)=0$. *Then the policy* $C^{\mathrm{O}}_{n-1}=D^{\mathrm{O}}(\varepsilon_{n-1})Z_{n-1}$ *yields the following value* E_n *for the expectation* $\mathbb{E}S_n$*:*

$$E_n=\sum_{j=1}^n\beta_{j-1}\ \text{ where }\ \beta_{j-1}=\mathbb{E}\Big\{\varphi(\varepsilon_{j-1},\varepsilon_j)\ln\Big[1+D^{\mathrm{O}}(\varepsilon_{j-1})g(\varepsilon_{j-1},\varepsilon_j)\Big]\Big\}. \tag{3.10}$$

Moreover, the value E_n gives the maximum of $\mathbb{E}S_n$ *over all strategies satisfying conditions* (a0)–(a3) *in* (3.6).

(d) *Suppose that the map $i \in M \mapsto D^{\rm O}(i)$ from assertion* (c) *is such that $D^{\rm O}(i) > 0$ (so the alternative* (i3A) *holds), $\forall\ i \in M$. Then the policy $C^{\rm O}_{n-1} = D^{\rm O}(\varepsilon_{n-1})Z_{n-1}$ maximises each summand $\overline{\alpha}_{j-1}$ in* (3.10)*, and therefore yields the maximum of the whole sum* $\mathbb{E}S_n$*, among strategies satisfying properties* (a0)–(a2) *in Eqn* (3.6).

Example 1.2: A two-state Markov chain. In the Markov case, when the trader observes the current state i, he/she uses the similar optimization procedure for the ith row of the 2×2 transition matrix $\mathtt{P} = (\mathtt{p}(i,k))$. Again suppose for simplicity that $M = \{0,1\}$, the WF $\varphi(i,j) \equiv 1$ and the RF g has $g(1) = -g(0) = \gamma > 0$. Also suppose that $b \in (0,1)$ is given. Then an analog of the previous picture emerges. Namely, set, for $i = 1,0$,

$$D^{\rm O}(i) = \begin{cases} \dfrac{\mathtt{p}(i,1) - \mathtt{p}(i,0)}{\gamma}, & \text{if } \dfrac{b}{2} \le \mathtt{p}(i,0) \le \mathtt{p}(i,1), \text{ and } \gamma \ge \mathtt{p}(i,1) - \mathtt{p}(i,0), \\ 0, & \text{otherwise.} \end{cases} \tag{3.11}$$

The policy $C^{\rm O}_n = D(\epsilon_n)Z_n(\boldsymbol{\varepsilon}_0^n)$ is optimal, under similar constraints. That is, if $D^{\rm O}(i) > 0$ for both $i = 0,1$ then the maximum is attained over strategies satisfying (a0) $C_j \in \mathfrak{W}_j$ and (a1,2) $0 \le C_j \le (1-b)Z_j$, $\forall\ j \ge 0$. Otherwise, if $D^{\rm O}(i) = 0$ for some i then it is among the strategies obeying (a0)–(a1,2) plus property (a3): $C_j(\boldsymbol{\varepsilon}_0^j)\big[\mathtt{p}(\varepsilon_n,1) - \mathtt{p}(\varepsilon_n,0)\big] = 0\ \ \forall\ j \ge 0$.

Viz., assume that the Markov chain (MC) is in the stationary regime, with stationary probabilities $\pi(1)$, $\pi(0)$. Then the maximal growth of $\mathbb{E}S_n$ is

$$\begin{aligned} E_n = n\Bigg(&\pi(1)\mathbf{1}\left(\frac{b}{2} \le \mathtt{p}(1,0) \le \mathtt{p}(1,1) \le \gamma + \mathtt{p}(1,0)\right) \\ &\times\Big\{\mathtt{p}(1,1)\ln\Big[1 + \mathtt{p}(1,1) - \mathtt{p}(1,0)\Big] + \mathtt{p}(1,0)\ln\Big[1 - \mathtt{p}(1,1) + \mathtt{p}(1,0)\Big]\Big\} \\ &+\pi(0)\mathbf{1}\left(\frac{b}{2} \le \mathtt{p}(0,0) \le \mathtt{p}(0,1) \le \gamma + \mathtt{p}(0,0)\right) \\ &\times\Big\{\mathtt{p}(0,1)\ln\Big[1 + \mathtt{p}(0,1) - \mathtt{p}(0,0)\Big] + \mathtt{p}(0,0)\ln\Big[1 - \mathtt{p}(0,1) + \mathtt{p}(0,0)\Big]\Big\}\Bigg). \end{aligned}$$

The last observation can be extended to general MCs. Indeed, suppose the trial MC starts with an invariant distribution $(\pi(i),\ i \in M)$. In this case the statements (c,d) of Theorem 1.1 assert that the maximum for $\mathbb{E}S_n$ is given as $n \sum\limits_{i,k \in M} \pi_i \mathtt{p}(i,k)\varphi(i,k)\ln\Big[1 + D^{\rm O}(i)g(i,k)\Big]$. Note that we do not need assumptions of irreducibility or aperiodicity: the invariant distribution is not assumed to be unique.

4 The cautious Kelly investments with a single riskless and several risky asset

The recursion for Z_n is similar to (3.1), with replacing scalar random variables by random vectors (RVs):

$$Z_n = Z_{n-1} + \underline{C}_{n-1} \cdot \underline{g}(\varepsilon_{n-1},\varepsilon_n) = Z_{n-1}\left[1 + \frac{\underline{C}_{n-1} \cdot \underline{g}(\varepsilon_{n-1},\varepsilon_n)}{Z_{n-1}}\right],\ n \ge 1. \tag{4.1}$$

Here and below,

$$\underline{C}_{n-1} \cdot \underline{g}(\varepsilon_{n-1},\varepsilon_n) = \sum_{s=1}^{K} C^{(s)}_{n-1} g^{(s)}(\varepsilon_{n-1},\varepsilon_n).$$

Also, $|\underline{C}_j| = \sum\limits_{s=1}^{K} C^{(s)}_j = \underline{C}_{n-1} \cdot \underline{1}$ where $\underline{1} = (1,\ldots,1)$, and we write $\underline{C}_j \ge \underline{0}$ if $C^{(s)}_j \ge 0\ \ \forall\ \ s$

As above, we set $S_n := \sum\limits_{j=1}^{n} \varphi(\varepsilon_{j-1},\varepsilon_j)\ln\dfrac{Z_j}{Z_{j-1}}$ and aim at maximizing the mean value $\mathbb{E}S_n$ in $\underline{C}_0,\ldots,\underline{C}_{n-1}$, under certain restrictions. Fix $b > 0$ and re-write the conditions outlining the portfolio classes under consideration: $\forall\ \ j \geq 0$,

$$\begin{aligned}
&\text{(a0)}\ \underline{C}_j \in \mathfrak{W}_j,\ \text{(predictability)},\ \text{(a1)}\ \underline{C}_j \geq \underline{0},\ |\underline{C}_j| < Z_j,\ \text{(sustainability)},\\
&\text{(a2)}\ 1 + \frac{\underline{C}_j \cdot \underline{g}(\varepsilon_j, \varepsilon_{j+1})}{Z_j} \geq b,\ \text{(no ruin), and}\\
&\text{(a3)}\quad \sum_{l \in M} \varphi(\varepsilon_j, l)\mathtt{q}_{j+1}(\varepsilon_j, l)\left[\underline{C}_j(\boldsymbol{\varepsilon}_0^j) \cdot \underline{g}(\varepsilon_j, l)\right] = 0\ \ \text{(weighted q,g-balance).}
\end{aligned}\tag{4.2}$$

We also assume, $\forall\ \ i \in M$, the (q–p) bound (3.7).

Consider the following conditions (i1)–(i4) which are vector counterparts of their scalar predecessors from (3.8). For convenience, we use the same labelling system as above.

(i) There exists a map $i \in M \mapsto \underline{D}(i)$ where vector $\underline{D}(i) = \big(D^{(1)}(i), \dots, D^{(K)}(i)\big)$ is such that $\forall$ $i, k \in M$,

$$\begin{aligned}
&\text{(i1)}\quad \underline{D}(i) \geq \underline{0},\ \text{ and }\ |\underline{D}(i)| < 1\quad \text{(D-sustainability)},\\
&\text{(i2)}\quad 1 + \underline{D}(i) \cdot \underline{g}(i,k) \geq b\ \text{ (D-no-ruin)},\\
&\text{(i3)}\quad \sum_{l \in M} \varphi(i,l)\mathtt{p}(i,l)\,\frac{\underline{D}(i) \cdot \underline{g}(i,l)}{1 + \underline{D}(i) \cdot \underline{g}(i,l)} = 0\ \text{(WE D,g-balance), and}\\
&\text{(i4)}\quad \text{the CFs }\ \mathtt{q}_j\ \text{ are }\ \mathtt{q}_j(i,k) = \frac{\mathtt{p}(i,k)}{1 + \underline{D}(i) \cdot \underline{g}(i,k)},\ j \geq 0\ \text{ (q-representation).}
\end{aligned}\tag{4.3}$$

Here the analog of the first alternative in (i3) is that $\underline{D}(i)$ satisfies a system of equations:

$$\begin{aligned}
&\text{(i3A)}\quad \sum_{l \in M} \mathtt{p}(i,l)\,\frac{\varphi(i,l) g^{(s)}(i,l)}{1 + \underline{D}(i) \cdot \underline{g}(i,l)} = 0\\
&\qquad\qquad \forall\ \ i \in M \text{ and } 1 \leq s \leq K\ \text{(strong WE D,g-balance).}
\end{aligned}\tag{4.4}$$

Cf. (i3A) in Eqn (3.8).

Theorem 3.1. *Assume the above setting* (4.1)–(4.4). *The following assertions hold true.*

(a) *Take any sequence* $\{\underline{C}_j,\ j \geq 0\}$ *obeying* (a0)–(a3) *in* (4.2). *Then the sequence* $\{S_n - \mathrm{A}_n,\ n \geq 1\}$ *is a supermartingale; hence* $\mathbb{E}\, S_n \leq \mathbb{E}\mathrm{A}_n\ \ \forall\ \ n \geq 1$.

(b) *To reach equality* $\mathbb{E}\, S_n = \sum\limits_{j=1}^{n} \mathbb{E}\alpha(\varepsilon_{j-1})$: *the sequence* $\{S_n - \mathrm{A}_n\}$ *is a martingale for a sequence of RVs* $\{\underline{C}_j\}$, *satisfying* (a0)–(a3) *iff the additional properties* (i), (ii) *below are fulfilled.*

(i) *There exists a map* $i \in M \mapsto \underline{D}(i)$ *where vector* $\underline{D}(i) = \big(D^{(1)}(i), \dots, D^{(K)}(i)\big)$ *is such that* $\forall$ $i, k \in M$, *properties* (i1)–(i3) *in* (4.3) *are fulfilled, and the CFs* $\mathtt{q}_j$ *are as in* (i4).

(ii) *The portfolio vectors* $\underline{C}_j$ *have components* $C_j^{(s)}(\boldsymbol{\varepsilon}_0^j) = D^{(s)}(\varepsilon_j)Z_j$, $1 \leq s \leq K$, $j \geq 0$. *That is, the prescribed fractions of the capital value* Z_j *are invested in the available returns.*

(c) *Suppose there exists a map* $i \in M \mapsto \underline{D}(i) = \big(D^{(1)}(i), \dots, D^{(K)}(i)\big)$ *fulfilling conditions* (i1)–(i3) *in Eqn* (4.3). *Let* $\mathbf{D}$ *stand for the array of values* $D^{(s)}(i)$, $i \in M$, $1 \leq s \leq K$, *and define the quantity* $E_n = E_n(\mathbf{D})$ *by*

$$E_n = \sum_{j=1}^{n} \beta_{j-1}\ \text{ where }\ \beta_{j-1} = \mathbb{E}\Big\{\varphi(\varepsilon_{j-1}, \varepsilon_j) \ln\, \Big[1 + \underline{D}(\varepsilon_{j-1}) \cdot \underline{g}(\varepsilon_{j-1}, \varepsilon_j)\Big]\Big\}.\tag{4.5}$$

Consider the optimization problem

$$\max\quad E_n(\mathbf{D})\quad \textit{subject to } \text{(i1)} - \text{(i3)}.\tag{4.6}$$

Let $\mathbf{D}^* = \arg\max E_n$ *be a (possibly, non-unique) optimizer, and* $E_n^* = E_n(\mathbf{D}^*)$ *denote the optimal value for* (4.6). *Then* E_n^* *defines the maximum of the expectation* $\mathbb{E}S_n$ *among all portfolios* $\{\underline{C}_j\}$ *satisfying the properties* (a0)–(a3) *in Eqn* (4.2). *The optimizer* $\mathbf{D}^*$ *written as a collection of vectors* $\underline{D}^*(i)$, $i \in M$, *yields a proportional investment portfolio where* $\underline{C}_j(\boldsymbol{\varepsilon}_0^j) = \underline{D}^*(\varepsilon_j)Z_j(\boldsymbol{\varepsilon}_j^j)$.

(d) *Suppose there exists a map* $i \in M \mapsto D(i)$ *fulfilling conditions* (i1)–(i2) *and* (i3A) *in Eqns* (4.3) *and* (4.4), *respectively. Then such a map is unique, and the proportional investment portfolio* $\underline{C}_{j-1}^{\mathrm{O}} = \underline{D}^*(\varepsilon_{j-1})Z_{j-1}$ *maximises each summand* β_{j-1} *in* (4.5). *Therefore, it yields the maximum of the whole sum* $\mathbb{E}S_n$, *among strategies satisfying properties* (a0)–(a2) *in Eqn* (4.2).

Finally, we study the general non-Markov trading with utility functions $\varphi(\epsilon_0^{j-1}, \epsilon_j)$**.**