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Limit Theorems for High Frequency Financial Data

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1 Introduction

In this lecture we focus on certain statistics ("power variation") of semimartingales of the form

$$X_t = X_0 + \int_0^t a_s \, ds + \int_0^t \sigma_s \, dW_s + J_t$$

where a is a drift term, σ is the volatility, W is a Brownian motion and J denotes the jumps, observed at high frequency, i.e. at

$$t_i = i \, \Delta_n \quad \text{with} \quad \Delta_n \to 0.$$

Typically the process $X = (X_t)_{t \geq 0}$ is observed on a fixed time intervall (say $[0, 1]$) and we are interested in analyzing the structure of the unobserved characteristics of X. The most important examples of such characteristics are:

(i) The quadratic variation

$$[X]_T = \int_0^T \sigma_s^2 \, ds + \sum_{0 \leq s \leq T} |\Delta X_s|^2$$

with $X_s = X_s - X_{s-}$.

(ii) The continuous part of the quadratic variation (in finance also called the "integrated volatility" or the "integrated variance")

$$[X]_t^c = \int_0^T \sigma_s^2 \, ds$$

and the discontinuous part of the quadratic variation

$$[X]_t^d = \sum_{0 \leq s \leq T} |\Delta X_s|^2$$

(iii) The Blumenthal-Getoor index $\beta \in [0, 2]$ of the process X. It gives information about the jump activity of $(J_t)_{t \geq 0}$.

The afore mentioned quantities are of huge importance in financial econometrics or mathematical finance.

We remark that under no arbitrage assumptions price processes must follow a semimartingale (see [8]). The estimation of quadratic variation using high frequency data is required for option pricing or risk management. It turns out that power variations, which are statistics of the form

$$\sum_{i=1}^{\lfloor T/\Delta_n \rfloor} |X_{i\Delta_n} - X_{(i-1)\Delta_n}|^p, \quad p \geq 0 \tag{1.1}$$

are very informative when analyzing the fine structure of X. In the last fifteen years there appeared a lot of publications on the asymptotic behaviour of power variations and related statistics. An important starting point was the work of Jacod ([9],[10]), who developed a first general (stable) central limit theorem for high frequency observations. Jean Jacod can be called the father of this research field.(See the recent book of Jacod and Protter [12].)

The aim of this lecture is twofold:
At first we will study the asymptotic behaviour of power variations

$$\sum_{i=1}^{\lfloor T/\Delta_n \rfloor} |X_{i\Delta_n} - X_{(i-1)\Delta_n}|^p, \quad p \geq 0$$

and related functionals. This icludes the law of large numbers and the central limit theorems.

Then we will apply the asymptotic results to estimation and testing problems. Some of the most important statistical problems are:

(i) Estimation of the quadratic variation

$$[X]_T = \int_0^T \sigma_s^2 \, ds + \sum_{0 \leq s \leq T} |\Delta X_s|^2$$

(ii) Robust estimation of the volatility functionals, i.e. integrated volatility

$$[X]_t^c = \int_0^T \sigma_s^2 \, ds$$

Here the notion of robustness refers to robustness of jumps.

(iii) Testing whether X has jumps.

(iv) Testing whether X has a Brownian component.

What can be estimated based on high frequency obervations $X_0, X_{\Delta_n}, X_{2\Delta_n}, \ldots, X_{\lfloor T/\Delta_n \rfloor \Delta_n}$?

(i) The drift process $(a_t)_{t\geq 0}$ can never be identified on a fixed time interval $[0, T]$ (unless $\sigma \equiv 0$)!

(ii) The volatility process $(\sigma_t)_{t\geq 0}$ can be consistently estimated.

(iii) The realised jumps ΔX_s can be identified.

(iv) But the law of jump part can not be identified.

2 A Crash Course on Stable Convergence and Mixed Normality

The power variations of semimartingales usually exhibit an asymptotic mixed normal distribution. For example when X is a continous semimartingale of the type

$$X_t = X_0 + \int_0^t a_s \, ds + \int_0^t \sigma_s \, dW_s,$$

we will show the following convergence in law for a fixed $T > 0$:

$$L_T^n = \Delta_n^{-1/2}\Big(\sum_{i=1}^{\lfloor T/\Delta_n \rfloor} |\Delta_i^n X|^2 - \int_0^T \sigma_s^2 \, ds \Big) \quad \xrightarrow{d} \quad L_T \sim MN\Big(0, 2\int_0^T \sigma_s^4 \, ds\Big)$$

where $\Delta_i^n X = X_{i\Delta_n} - X_{(i-1)\Delta_n}$. Here the limiting variable L_T has a mixed normal distribution with mean 0 and conditional variance $2\int_0^T \sigma_s^4 \, ds$. This means

$$L_T \stackrel{d}{=} \Big(2\int_0^T \sigma_s^4 \, ds\Big)^{1/2} U, \quad U \sim N(0,1)$$

with U being independent of $(\sigma_t)_{t\geq 0}$. Notice that the characteristic function of L_T is given by

$$\mathbb{E}[(itL_T)] = \mathbb{E}\Big[\exp\Big(-t\int_0^T \sigma_s^4 \, ds\Big)\Big].$$

In order to obtain confidence regions for the integrated volatility $\int_0^T \sigma_s^2 \, ds$, we will also show that

$$V_n^2 := \frac{2}{3\Delta_n} \sum_{i=1}^{\lfloor T/\Delta_n \rfloor} |\Delta_i^n X|^4 \quad \xrightarrow{\mathbb{P}} \quad V^2 = 2\int_0^T \sigma_s^4 \, ds.$$

In the second step we would like to conclude that

$$\frac{L_t^n}{V_n} \xrightarrow{d} N(0,1).$$

However, the weak convergence $L_t^n \xrightarrow{d} L$ does not imply the joint convergence $(L_T^n, V_n) \xrightarrow{d} (L_T, V)$, which is required to conclude the above statement.
For this reason we require a stronger mode of convergence than convergence in law. Stable convergence turns out to be an exactly right type of convergence to guarantee the afore mentioned statement. In the following subsection we will give a formal definition of stable convergence and derive its most useful properties.

2.1 Stable Convergence

In this subsection all random variables or processes are defined on some probability space $(\Omega, \mathbb{F}, \mathbb{P})$. We start with a definition of stable convergence.

Definition 2.1
Let Y_n be a sequence of random variables with values in a Polish space $(E, \mathcal{E})$. We say that Y_n converges stably with limit Y, written $Y_n \xrightarrow{st} Y$, where Y is defined on an extension $(\Omega', \mathbb{F}', \mathbb{P}')$ of the original probability space $(\Omega, \mathbb{F}, \mathbb{P})$, iff for any bounded, continuous function g and any bounded $\mathbb{F}$-measurable random variable Z it holds that

$$\mathbb{E}[g(Y_n)Z] \to \mathbb{E}'[g(Y)Z] \quad \text{as} \quad n \to \infty. \tag{2.1}$$

First of all, we remark that random variables Y_n in the above definition can also be random processes. We immediately see that stable convergence is a stronger mode of convergence than weak convergence (which corresponds to $Z = 1$), but weaker than convergence in probability.
For the sake of simplicity we will only deal with stable convergence of $\mathbb{R}^d$-valued random variables in this subsection. The next proposition gives a much simpler characterization of stable convergence which is closer to the original definition of Rény in [15] (see also [2]).

Proposition 2.1
The following properties are equivalent:

(i) $Y_n \xrightarrow{st} Y$

(ii) $(Y_n, Z) \xrightarrow{d} (Y, Z)$ *for any $\mathbb{F}$-measurable variable Z*

(iii) $(Y_n, Z) \xrightarrow{st} (Y, Z)$ *for any $\mathbb{F}$-measurable variable Z*

The assertion of Proposition 2.1 is shown via the usual approximation techniques and we leave the details to the reader.
For the moment it is not quite clear why an extension of the original probability space $(\Omega, \mathbb{F}, \mathbb{P})$ in Definition 2.1 is required. The next lemma gives the answer.

Lemma 2.1
Assume that $Y_n \xrightarrow{st} Y$ and Y is $\mathbb{F}$-measurable. Then

$$Y_n \xrightarrow{\mathbb{P}} Y.$$

Proof: As $Y_n \xrightarrow{st} Y$ and Y is $\mathbb{F}$-measurable, we deduce by Proposition 2.1(ii) that

$$(Y_n, Y) \xrightarrow{d} (Y, Y).$$

Hence, $Y_n - Y \xrightarrow{d} 0$, and $Y_n \xrightarrow{\mathbb{P}} Y$ readily follows. □

Lemma 2.1 tells us that the extension of the original probability space is not required iff we have $Y_n \xrightarrow{\mathbb{P}} Y$. But if we have "real" stable convergence $Y_n \xrightarrow{st} Y$, what type of extension usually appears? A partial answer is given in the following example.

Example 2.1
Let $(X_i)_{i\geq 1}$ be a sequence of i.i.d. random variables with $\mathbb{E}[X_1] = 0$ and $\mathbb{E}[X_1^2] = 1$, defined on $(\Omega, \mathbb{F}, \mathbb{P})$. Assume that $\mathbb{F} = \sigma(X_1, X_2, \ldots)$. Setting $Y_n = \frac{1}{\sqrt{n}} \sum_{i=1}^n X_i$ we obtain that

$$Y_n \xrightarrow{d} Y \sim N(0,1),$$

which is of course a well-known result. Is there a stable version of this weak convergence? The answer is yes. Let $Y \sim N(0,1)$ be independent of $\mathbb{F}$. Then

$$Y_n \xrightarrow{st} Y.$$

This can be shown as follows. For any collection $t_1, \ldots, t_k \in \mathbb{N}$, we deduce that

$$(Y_n, X_{t_1}, \ldots, X_{t_k}) \xrightarrow{d} (Y, X_{t_1}, \ldots, X_{t_k})$$

as Y_n is asymptotically independent of $(X_{t_1}, \ldots, X_{t_k})$ and Y is independent of $\mathbb{F}$. This implies that

$$(Y_n, Z) \xrightarrow{d} (Y, Z)$$

for any $\mathbb{F}$-measurable Z (since $\mathbb{F} = \sigma(X_1, X_2, \ldots)$). This implies that $Y_n \xrightarrow{st} Y$. □

In fact, the described situation is pretty typical. Usually, we only require a new standard normal variable that is independent of $\mathbb{F}$. Thus, the extension is given by a product space. However, more complicated extensions may appear.

In the last proposition of this subsection we present the delta method for stable convergence.

Proposition 2.2
Let Y_n, V_n, Y, X, V be $\mathbb{R}^d$-valued random variables and let $g : \mathbb{R}^d \to \mathbb{R}$ be a C^1-function.

(i) If $Y_n \xrightarrow{st} Y$ and $V_n \xrightarrow{\mathbb{P}} V$ then $(Y_n, V_n) \xrightarrow{st} (Y, V)$.

(ii) Let $d = 1$ and $Y_n \xrightarrow{st} Y \sim MN(0, V^2)$ with V being $\mathbb{F}$-measurable. Assume that $V_n \xrightarrow{\mathbb{P}} V$ and $V_n, V > 0$. Then

$$\frac{Y_n}{V_n} \xrightarrow{d} N(0,1).$$

(iii) Let $\sqrt{n}(Y_n - Y) \xrightarrow{st} X$. Then $\sqrt{n}(g(Y_n) - g(Y)) \xrightarrow{st} \nabla g(Y) X$.

Proof:

(i) $Y_n \xrightarrow{st} Y$ implies $(Y_n, V) \xrightarrow{d} (Y, V)$. Since $V_n \xrightarrow{\mathbb{P}} V$, we also have $(Y_n, V_n) \xrightarrow{d} (Y, V)$.

(ii) We know from part (i) that $(Y_n, V_n) \xrightarrow{d} (Y, V)$. Applying the continuous mapping theorem with $f(x, z) = \frac{x}{z}$ to (Y_n, V_n) we deduce that

$$\frac{Y_n}{V_n} \xrightarrow{d} \frac{Y}{V} \sim N(0,1).$$

(iii) Since $\sqrt{n}(Y_n - Y) \xrightarrow{st} X$ we have

$$\|Y_n - Y\| \xrightarrow{\mathbb{P}} 0.$$

The mean value theorem implies that

$$\sqrt{n}(g(Y_n) - g(Y)) = \sqrt{n}\nabla g(\xi_n)(Y_n - Y)$$

for some ξ_n with $\|\xi_n - Y\| \leq \|Y_n - Y\|$. Clearly, $\xi_n \xrightarrow{\mathbb{P}} Y$. Thus, by part (i) we obtain $(\xi_n, \sqrt{n}(Y_n - Y)) \xrightarrow{st} (Y, X)$. This implies the assertion by continuous mapping because ∇g is continuous. □

2.2 Jacod's Stable Central Limit Theorem

In practice it is difficult to prove stable convergence, especially for processes. As for weak convergence, it is sufficient to show stable convergence of the finite dimensional distributions and tightness. However, proving stable convergence of the finite dimensional distributions is not easy, because the structure of the σ-algebra $\mathbb{F}$ can be rather complicated.
Jacod (see [10]) has derived a general stable central limit theorem for partial sums of triangular arrays. Below we assume that all processes are defined on the filtered probability space $(\Omega, \mathbb{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$. We consider functionals of the form

$$Y_t^n = \sum_{i=1}^{[t/\Delta_n]} X_{in}, \tag{2.2}$$

where the X_{in}'s are $\mathcal{F}_{i\Delta_n}$-measurable and square integrable random variables. Moreover, we assume that X_{in}'s are "fully generated" by a Brownian motion W. Recall that power variations are statistics of the type (2.2).

Before we present the main theorem of this subsection, we need to introduce some notations. Below, $([M, N]_s)_{s\geq 0}$ denotes the covariation process of two (one-dimensional) semimartingales $(M_s)_{s\geq 0}$ and $(N_s)_{s\geq 0}$. We write $V^n \xrightarrow{\text{u.c.p.}} V$ whenever

$$\sup_{t\in[0,T]} |V_t^n - V_t| \xrightarrow{\mathbb{P}} 0 \quad \forall\, T > 0.$$

Theorem 2.1 (Jacod's Theorem (1997))
Assume there exist absolutely continuous processes F, G, and a continuous process B

with finite variation such that the following conditions are satisfied for each $t \in [0,T]$:

$$\sum_{i=1}^{[t/\Delta_n]} \mathbb{E}[X_{in}|\mathcal{F}_{(i-1)\Delta_n}] \xrightarrow{u.c.p.} B_t, \tag{2.3}$$

$$\sum_{i=1}^{[t/\Delta_n]} \Big(\mathbb{E}[X_{in}^2|\mathcal{F}_{(i-1)\Delta_n}] - \mathbb{E}^2[X_{in}|\mathcal{F}_{(i-1)\Delta_n}]\Big) \xrightarrow{\mathbb{P}} F_t = \int_0^t (v_s^2 + w_s^2) ds, \tag{2.4}$$

$$\sum_{i=1}^{[t/\Delta_n]} \mathbb{E}[X_{in}\Delta_i^n W|\mathcal{F}_{(i-1)\Delta_n}] \xrightarrow{\mathbb{P}} G_t = \int_0^t v_s ds, \tag{2.5}$$

$$\sum_{i=1}^{[t/\Delta_n]} \mathbb{E}[X_{in}^2 1_{\{|X_{in}>\varepsilon|\}}|\mathcal{F}_{(i-1)\Delta_n}] \xrightarrow{\mathbb{P}} 0 \qquad \forall \varepsilon > 0, \tag{2.6}$$

$$\sum_{i=1}^{[t/\Delta_n]} \mathbb{E}[X_{in}\Delta_i^n N|\mathcal{F}_{(i-1)\Delta_n}] \xrightarrow{\mathbb{P}} 0, \tag{2.7}$$

where $(v_s)_{s\geq 0}$ and $(w_s)_{s\geq 0}$ are predictable processes and condition (2.7) holds for all bounded $\mathcal{F}_t$-martingales $(N_s)_{s\geq 0}$ with $N_0 = 0$ and $[W,N]_s \equiv 0$. Then we obtain the stable convergence of processes:

$$Y_t^n \xrightarrow{st} Y_t = B_t + \int_0^t v_s dW_s + \int_0^t w_s dW_s', \tag{2.8}$$

on $D([0,T])$, where W' is a Brownian motion defined on an extension of the original probability space $(\Omega, \mathbb{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ and independent of the original σ-algebra $\mathbb{F}$.

Remark 2.1
Let us shortly comment on the conditions of Theorem 2.1.

(i) Condition (2.3) determines the drift (or bounded variation part) of the limiting process Y.

(ii) Condition (2.4) determines the quadratic variation of Y.

(iii) Condition (2.6) ("Lindeberg condition") ensures that the limiting process Y has no jump part.

(iv) Condition (2.8) implies that on the original probability space $(\Omega, \mathbb{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ there is only one martingale W, which contributes to Y. The combination of (2.4) and (2.5) identifies the distribution of the quadratic variation of Y between the dW_s and dW_s' part. □

Remark 2.2
At the moment Theorem 2.1 is a probabilistic result that has no statistical applications in general, because the distribution of Y is unknown. However, when $B \equiv 0$ and $v \equiv 0$,

which is the case for the most interesting situations, things become different! We remark that, for any fixed $t > 0$,

$$\int_0^t w_s dW_s' \sim MN\Big(0, \int_0^t w_s^2 ds\Big),$$

since W' is independent of $\mathbb{F}$. Hence

$$\frac{Y_t^n}{\sqrt{\int_0^t w_s^2 ds}} \xrightarrow{d} N(0,1),$$

and the convergence still holds true if we replace the denominator by a consistent estimator. The latter can be applied to obtain confidence bands or to solve other statistical problems. □

Next, we will apply Theorem 2.1 to a particular example.

Example 2.2
Let σ be a càdlàg, $\mathcal{F}_t$-adapted and bounded process and let $g, h : \mathbb{R} \to \mathbb{R}$ be continuous functions, where h satisfies the polynomial growth condition $|h(x)| < C(1 + |x|^r)$ for some $r > 0$ and $C > 0$. Define

$$Y_t^n = \sum_{i=1}^{[t/\Delta_n]} X_{in}, \qquad X_{in} = \Delta_n^{1/2} g(\sigma_{(i-1)\Delta_n})\Big(h\Big(\frac{\Delta_i^n W}{\sqrt{\Delta_n}}\Big) - \mathbb{E}\Big[h\Big(\frac{\Delta_i^n W}{\sqrt{\Delta_n}}\Big)\Big]\Big). \tag{2.9}$$

Note that the X_{in}'s have a pretty simple structure, since $\Delta_i^n W$ is independent of $\mathcal{F}_{(i-1)\Delta_n}$ and $\Delta_i^n W/\sqrt{\Delta_n} \sim N(0,1)$. Now we need to check the conditions (2.3) - (2.7) from Theorem 2.1.

(2.3): $\mathbb{E}[X_{in}|\mathcal{F}_{(i-1)\Delta_n}] = 0 \quad \Longrightarrow \quad$ B$\equiv 0$. □

(2.4), (2.5):

$$\mathbb{E}[X_{in}^2|\mathcal{F}_{(i-1)\Delta_n}] - \mathbb{E}^2[X_{in}|\mathcal{F}_{(i-1)\Delta_n}] = \Delta_n \, g^2(\sigma_{(i-1)\Delta_n}) \, \text{var}(h(U)),$$

with $U \sim N(0,1)$

$$\mathbb{E}[X_{in}\Delta_i^n W|\mathcal{F}_{(i-1)\Delta_n}] = \Delta_n \, g(\sigma_{(i-1)\Delta_n})\mathbb{E}[h(U)U]$$

And so it follows that

$$F_t = a^2 \int_0^t g^2(\sigma_s) ds \quad \text{with } a^2 = \text{var}(h(U)),$$

$$G_t = b \int_0^t g(\sigma_s) ds \quad \text{with } b = \mathbb{E}[h(U)U].$$

Thus, we can set $w_s = \sqrt{a^2 - b^2}\, g(\sigma_s)$, $v_s = b\, g(\sigma_s)$ in (2.4) and (2.5). □

(2.6):

$$\sum_{i=1}^{[t/\Delta_n]} \mathbb{E}[X_{in}^2 1_{\{|X_{in}|>\varepsilon\}}|\mathcal{F}_{(i-1)\Delta_n}] \leq \varepsilon^{-2} \sum_{i=1}^{[t/\Delta_n]} \mathbb{E}[X_{in}^4|\mathcal{F}_{(i-1)\Delta_n}] \leq C\frac{\Delta_n}{\varepsilon^2} \longrightarrow 0$$

for some $C > 0$, because σ is bounded. This implies (2.6). □

(2.7): Itô-Clark representation implies that

$$h\Big(\frac{\Delta_i^n W}{\sqrt{\Delta_n}}\Big) - \mathbb{E}\Big[h\Big(\frac{\Delta_i^n W}{\sqrt{\Delta_n}}\Big)\Big] = \int_{(i-1)\Delta_n}^{i\Delta_n} \eta_s^n dW_s.$$

for some process $(\eta_s^n)_{s\geq 0}$. We obtain that

$$\begin{aligned} \mathbb{E}[X_{in}\Delta_i^n N|\mathcal{F}_{(i-1)\Delta_n}] &= \Delta_n^{1/2} g(\sigma_{(i-1)\Delta_n})\, \mathbb{E}\Big[\int_{(i-1)\Delta_n}^{i\Delta_n} \eta_s^n dW_s \int_{(i-1)\Delta_n}^{i\Delta_n} dN_s\Big] \\ &= \Delta_n^{1/2} g(\sigma_{(i-1)\Delta_n})\, \mathbb{E}\Big[\int_{(i-1)\Delta_n}^{i\Delta_n} \eta_s^n d\underbrace{[W,N]}_{=0}{}_s\Big] = 0 \end{aligned}$$

Putting things together we deduce that

$$Y_t^n \xrightarrow{st} Y_t = b\int_0^t g(\sigma_s)dW_s + \sqrt{a^2-b^2}\int_0^t g(\sigma_s)dW_s'.$$

Furthermore, when h is an even function then $b = 0$ and we have

$$Y_t^n \xrightarrow{st} Y_t = a\int_0^t g(\sigma_s)dW_s' \quad \sim \quad MN\Big(0, a^2\int_0^t g^2(\sigma_s)ds\Big)$$

□

3 Limit Theorems for Continuous Semimartingales

In this section we consider continuous semimartingales of the form

$$X_t = X_0 + \int_0^t a_s ds + \int_0^t \sigma_s dW_s, \tag{3.1}$$

where $(a_s)_{s\geq 0}$ is a càglàd process and $(\sigma_s)_{s\geq 0}$ is a càdlàg, adapted process. Recall that
càglàd $\hat{=}$ left continuous with right limits
càdlàg $\hat{=}$ right continuous with left limits
$(W_s)_{s\geq 0}$ is a 1-dimensional Brownian motion. All processes are defined on a filtered probability space $(\Omega, \mathbb{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$. We consider high frequency statistics of the form

$$V(f)_t^n = \Delta_n \sum_{i=1}^{[t/\Delta_n]} f\Big(\frac{\Delta_i^n X}{\sqrt{\Delta_n}}\Big), \qquad \Delta_i^n X = X_{i\Delta_n} - X_{(i-1)\Delta_n}. \tag{3.2}$$

Notice that for $f(x) = |x|^p$ we obtain the power variation of X. We start with the "law of large numbers" for $V(f)_t^n$. For any function $f : \mathbb{R} \to \mathbb{R}$, we define

$$\rho_x(f) = \mathbb{E}[f(xU)], \tag{3.3}$$

for $x \in \mathbb{R}$ and $U \sim N(0,1)$, if the above expectation exitst.

Theorem 3.1
Assume that the function f is continuous and has polynomial growth, i.e. $|f(x)| \leq C(1 + |x|^p)$ for some $C > 0$ and $p \geq 0$. Then

$$V(f)_t^n \xrightarrow{\text{u.c.p.}} V(f)_t = \int_0^t \rho_{\sigma_s}(f) ds. \tag{3.4}$$

Recall that $V(f)_t^n \xrightarrow{\text{u.c.p.}} V(f)_t$ stands for $\sup_{t\in[0,T]} |V(f)_t^n - V(f)_t| \xrightarrow{\mathbb{P}} 0$ for all $T > 0$.
We remark that the drift process $(a_s)_{s\geq 0}$ does not influence the limit $V(f)_t$. We will see later why this phenomenon appears. Next, we present an important application of Theorem 3.1.

Example 3.1 (Power variation)
As we mentioned, the case $f(x) = |x|^p$ corresponds to the subclass of power variations. It is the most important subclass of statistics in financial econometrics. For $f(x) = |x|^p$,

Theorem 3.1 translates to

$$V(f)_t^n \xrightarrow{u.c.p.} V(f)_t = m_p \int_0^t |\sigma_s|^p ds,$$

with $m_p = \mathbb{E}[|N(0,1)|^p]$ *since* $\rho_x(f) = m_p|x|^p$. *For* $f(x) = x^2$ *we recover a well-known result from stochastic calculus*

$$V(f)_t^n = \sum_{i=1}^{[t/\Delta_n]} |\Delta_i^n X|^2 \xrightarrow{u.c.p.} [X]_t = \int_0^t \sigma_s^2 ds.$$

□

Sketch of the proof of Theorem 3.1.

(i) *The crucial approximation:* First of all, observe that

$$\Delta_i^n X = \underbrace{\int_{(i-1)\Delta_n}^{i\Delta_n} a_s ds}_{=O_p(\Delta_n)} + \underbrace{\int_{(i-1)\Delta_n}^{i\Delta_n} \sigma_s dW_s}_{=O_p(\Delta_n^{1/2})}, \tag{3.5}$$

where the second approximation follows by *Burkholder's inequality* (see e.g. Theorem IV.4.1 in [16]):

$$\mathbb{E}\Big[\Big|\int_a^b \sigma_s\, dW_s\Big|^p\Big] \leq C_p\, \mathbb{E}\Big[\Big|\int_a^b \sigma_s^2\, dW_s\Big|^{p/2}\Big] \tag{3.6}$$

for all $p \geq 0$, if the second expectation exists. Thus, the influence of the drift process $(a_s)_{s\geq 0}$ is negligible for the first order asymptotics. Indeed, we have

$$\frac{\Delta_i^n X}{\sqrt{\Delta_n}} \approx \alpha_i^n = \Delta_n^{-1/2} \sigma_{(i-1)\Delta_n} \Delta_i^n W, \tag{3.7}$$

which is the crucial approximation for proving all asymptotic results. Note that, compared to X, the α_i^n's have a very simple structure: They are uncorrelated and $\alpha_i^n \sim MN(0, \sigma_{(i-1)\Delta_n}^2)$. It holds that

$$\mathbb{E}[f(\alpha_i^n)|\mathcal{F}_{(i-1)\Delta_n}] = \rho_{\sigma_{(i-1)\Delta_n}}(f),$$

which explains the definition of $\rho_x(f)$. □

(ii) *From local boundedness to boundedness:* Since the processes $(a_s)_{s\geq 0}$ and $(\sigma_{s-})_{s\geq 0}$ are assumed to be càglàd, they are locally bounded, i.e. there exists an increasing sequence of stopping times T_k with $T_k \xrightarrow{\text{a.s.}} \infty$ such that

$$|a_s| + |\sigma_{s-}| \leq C_k, \qquad \forall s \leq T_k$$

for all $k \geq 1$. Using this fact it is indeed possible to assume w.l.o.g. that $(a_s)_{s\geq 0}$, $(\sigma_{s-})_{s\geq 0}$ are bounded, because Theorem 3.1 is stable under stopping. To illustrate these ideas set $a_s^{(k)} = a_s 1_{\{s\leq T_k\}}$, $\sigma_s^{(k)} = \sigma_s 1_{\{s<T_k\}}$. Note that the processes $a^{(k)}$, $\sigma^{(k)}$ are bounded for all $k \geq 1$. Associate $X^{(k)}$ with $a^{(k)}$, $\sigma^{(k)}$ by (3.1), $V^{(k)}(f)_t^n$ with $X^{(k)}$ by (3.2) and $V^{(k)}(f)_t$ with $\sigma^{(k)}$ by (3.4). Now, notice that

$$X_t^{(k)} = X_t, \qquad V^{(k)}(f)_t^n = V(f)_t^n, \qquad V^{(k)}(f)_t = V(f)_t, \qquad \forall t \leq T_k.$$

As $T_k \xrightarrow{\text{a.s.}} \infty$ it is sufficient to prove $V^{(k)}(f)_t^n \xrightarrow{\text{u.c.p.}} V^{(k)}(f)_t$ for each $k \geq 1$. For this reason we may assume w.l.o.g. that $(a_s)_{s\geq 0}$, $(\sigma_{s-})_{s\geq 0}$ are bounded in (ω, t). □

(iii) *Main step:* Since f is continuous and σ is càdlàg (and bounded w.l.o.g.), it is relatively simple to show that

$$V(f)_t^n - \Delta_n \sum_{i=1}^{[t/\Delta_n]} f(\alpha_i^n) \xrightarrow{\text{u.c.p.}} 0. \tag{3.8}$$

On the other hand, it holds that

$$\Delta_n \sum_{i=1}^{[t/\Delta_n]} \mathbb{E}[f(\alpha_i^n)|\mathcal{F}_{(i-1)\Delta_n}] = \Delta_n \sum_{i=1}^{[t/\Delta_n]} \rho_{\sigma_{(i-1)\Delta_n}}(f) \xrightarrow{\text{u.c.p.}} V(f)_t.$$

Hence, we are left to proving the convergence

$$\Delta_n \sum_{i=1}^{[t/\Delta_n]} \left\{ f(\alpha_i^n) - \mathbb{E}[f(\alpha_i^n)|\mathcal{F}_{(i-1)\Delta_n}] \right\} \xrightarrow{\text{u.c.p.}} 0$$

But this follows directly from

$$\Delta_n^2 \sum_{i=1}^{[t/\Delta_n]} \mathbb{E}[f^2(\alpha_i^n)|\mathcal{F}_{(i-1)\Delta_n}] = \Delta_n^2 \sum_{i=1}^{[t/\Delta_n]} \rho_{\sigma_{(i-1)\Delta_n}}(f) \xrightarrow{\mathbb{P}} 0.$$

Hence

$$V(f)_t^n \xrightarrow{\text{u.c.p.}} V(f)_t.$$

□

Now we turn our attention to the stable central limit theorem associated with Theorem 3.1. Here we require a stronger assumption on the volatility process σ to be able to deal with the approximation error induced by (3.7). More precisely, the process σ is also a continuous semimartingale of the form:

$$\sigma_t = \sigma_0 + \int_0^t \tilde{a}_s ds + \int_0^t \tilde{\sigma}_s dW_s + \int_0^t \tilde{\tau}_s dV_s, \tag{3.9}$$

where the processes $(\tilde{a}_s)_{s\geq 0}$, $(\tilde{\sigma}_s)_{s\geq 0}$, $(\tilde{\tau}_s)_{s\geq 0}$ are càdlàg adapted and V is a Brownian motion independent of W.

In fact, the assumption (3.9) is motivated by econometric applications, as it is satisfied for many stochastic volatility models. Next, for any function $f : \mathbb{R} \to \mathbb{R}$ and $k \in \mathbb{N}$, we define

$$\rho_x(f,k) = \mathbb{E}[f(xU)U^k], \qquad U \sim N(0,1). \tag{3.10}$$

Note that $\rho_x(f) = \rho_x(f,0)$.

Theorem 3.2
Assume that $f \in C^1(\mathbb{R})$ with f, f' having polynomial growth and that condition (3.9) is satisfied. Then the stable convergence of processes

$$\Delta_n^{-1/2}\Big(V(f)_t^n - V(f)_t\Big) \xrightarrow{st} L(f)_t = \int_0^t b_s ds + \int_0^t v_s dW_s + \int_0^t w_s dW'_s, \tag{3.11}$$

holds, where

$$\begin{aligned} b_s &= a_s \rho_{\sigma_s}(f') + \frac{1}{2}\tilde{\sigma}_s(\rho_{\sigma_s}(f',2) - \rho_{\sigma_s}(f')), \\ v_s &= \rho_{\sigma_s}(f,1), \\ w_s &= \sqrt{\rho_{\sigma_s}(f^2) - \rho^2_{\sigma_s}(f) - \rho^2_{\sigma_s}(f,1)} \end{aligned}$$

and W' is a Brownian motion defined on an extension of the original probability space $(\Omega, \mathbb{F}, (\mathcal{F}_t)_{t\ge 0}, \mathbb{P})$ and independent of the original σ-algebra $\mathbb{F}$.

As a consequence of Theorem 3.2 we obtain a simple but very important lemma.

Lemma 3.1
Assume that $f : \mathbb{R} \to \mathbb{R}$ is an even function and that the conditions of Theorem 3.2 hold. Then $\rho_x(f') = \rho_x(f',2) = \rho_x(f,1) = 0$, and we deduce that

$$\Delta_n^{-1/2}\Big(V(f)_t^n - V(f)_t\Big) \xrightarrow{st} L(f)_t = \int_0^t w_s dW'_s,$$

with $w_s = \sqrt{\rho_{\sigma_s}(f^2) - \rho^2_{\sigma_s}(f)}$.

As we mentioned in Remark 2.2, $L(f)_t$ has a mixed normal distribution (for any $t > 0$) when f is an even function. Indeed, this is the case for almost all statistics used in practice. Let us now return to Example 3.1.

Example 3.2 (Power variations)
We consider again the class of functions $f(x) = |x|^p$ ($p > 0$), which are obviously even. By Lemma 3.1 we deduce that

$$\Delta_n^{-1/2}\Big(V(f)_t^n - m_p \int_0^t |\sigma_s|^p\Big) \xrightarrow{st} L(f)_t = \sqrt{m_{2p} - m_p^2} \int_0^t |\sigma_s|^p dW'_s. \tag{3.12}$$

In fact, the above convergence can be deduced from Lemma 3.1 only for $p > 1$, since otherwise $f(x) = |x|^p$ is not differentiable at 0. However, it is possible to extend the theory to the case $0 < p \leq 1$ under a further condition on σ; see BGJPS (06). By Theorem 3.1 and Proposition 2.2 we are able to derive a feasible *version of Lemma 3.1 associated with $f(x) = |x|^p$:*

$$\frac{\Delta_n^{-1/2}\Big(\Delta_n^{1-p/2}|\Delta_i^n X|^p - m_p \int_0^t |\sigma_s|^p \, ds\Big)}{\sqrt{\frac{m_{2p}-m_p^2}{m_{2p}} V(f^2)_t^n}} \xrightarrow{d} N(0,1),$$

which can be used for statistical purposes. For the case of quadratic variation, i.e. $f(x) = x^2$, this translates to

$$\frac{\Delta_n^{-1/2}\Big(\sum_{i=1}^{[t/\Delta_n]} |\Delta_i^n X|^2 - \int_0^t \sigma_s^2\Big)}{\sqrt{\frac{2}{3}\Delta_n^{-1}\sum_{i=1}^{[t/\Delta_n]} |\Delta_i^n X|^4}} \xrightarrow{d} N(0,1).$$

Quite interestingly, the stable convergence for the case of quadratic variation can be proved without imposing the condition (3.9); this is not possible for other powers p. $\square$

Sketch of the proof of Theorem 3.2:
(i) *CLT for the approximation (3.7):* First of all, we observe that Theorem 3.2 is also stable under stopping. Thus, we can assume w.l.o.g. that the processes $(a_s)_{s\geq 0}$, $(\sigma_s)_{s\geq 0}$, $(\tilde{a}_s)_{s\geq 0}$, $(\tilde{\sigma}_s)_{s\geq 0}$, $(\tilde{\tau}_s)_{s\geq 0}$ are bounded. In a first step, we show the central limit theorem for the approximation

$$\alpha_i^n = \Delta_n^{-1/2}\sigma_{(i-1)\Delta_n}\Delta_i^n W.$$

More precisely, we want to prove that

$$\sum_{i=1}^{[t/\Delta_n]} X_{in} \xrightarrow{st} \int_0^t v_s dW_s + \int_0^t w_s dW'_s, \qquad X_{in} = \Delta_n^{1/2}\Big(f(\alpha_i^n) - \mathbb{E}[f(\alpha_i^n)|\mathcal{F}_{(i-1)\Delta_n}]\Big),$$

where the processes $(v_s)_{s\geq 0}$ and $(w_s)_{s\geq 0}$ are defined in Theorem 3.2. In principle, we can follow the ideas of Example 2.2: we immediately deduce the convergence

$$\sum_{i=1}^{[t/\Delta_n]} \mathbb{E}[X_{in}^2|\mathcal{F}_{(i-1)\Delta_n}] \xrightarrow{\mathbb{P}} F_t = \int_0^t (\rho_{\sigma_s}(f^2) - \rho_{\sigma_s}^2(f))ds,$$

$$\sum_{i=1}^{[t/\Delta_n]} \mathbb{E}[X_{in}\Delta_i^n W|\mathcal{F}_{(i-1)\Delta_n}] \xrightarrow{\mathbb{P}} G_t = \int_0^t \rho_{\sigma_s}(f,1)ds.$$

On the other hand, conditions (2.3) with $B \equiv 0$, (2.6) and (2.7) of Theorem 2.1 are shown as in Example 2.2. Consequently, we deduce that

$$\sum_{i=1}^{[t/\Delta_n]} X_{in} \xrightarrow{st} \int_0^t v_s dW_s + \int_0^t w_s dW'_s.$$

□

(*ii*) *CLT for the canonical process:* Before we proceed with the proof of Theorem 3.2 we need to present a further intermediate step. In fact, it is much more natural to consider a central limit theorem for the "canonical process"

$$L(f)_t^n = \Delta_n^{1/2} \sum_{i=1}^{[t/\Delta_n]} \Big\{ f\Big(\frac{\Delta_i^n X}{\sqrt{\Delta_n}}\Big) - \mathbb{E}\Big[f\Big(\frac{\Delta_i^n X}{\sqrt{\Delta_n}}\Big)\Big|\mathcal{F}_{(i-1)\Delta_n}\Big]\Big\}$$

since the latter is a martingale. Notice that

$$L(f)_t^n - \sum_{i=1}^{[t/\Delta_n]} X_{in} = \sum_{i=1}^{[t/\Delta_n]} \Big(Y_{in} - \mathbb{E}\big[Y_{in}\big|\mathcal{F}_{(i-1)\Delta_n}\big]\Big)$$

with

$$Y_{in} = \Delta_n^{1/2}\Big(f\Big(\frac{\Delta_i^n X}{\sqrt{\Delta_n}}\Big) - f(\alpha_i^n)\Big).$$

Since

$$\sum_{i=1}^{[t/\Delta_n]} \mathbb{E}\big[Y_{in}^2\big|\mathcal{F}_{(i-1)\Delta_n}\big] = \Delta_n \sum_{i=1}^{[t/\Delta_n]} \mathbb{E}\Big[\Big(f\Big(\frac{\Delta_i^n X}{\sqrt{\Delta_n}}\Big) - f(\alpha_i^n)\Big)^2\Big|\mathcal{F}_{(i-1)\Delta_n}\Big] \xrightarrow{\mathbb{P}} 0$$

(this is shown as in (3.8)), we conclude that

$$L(f)_t^n - \sum_{i=1}^{[t/\Delta_n]} X_{in} \xrightarrow{\text{u.c.p.}} 0,$$

Hence,

$$L(f)_t^n \xrightarrow{st} \int_0^t v_s dW_s + \int_0^t w_s dW_s'.$$

□

(*iii*) *The final step:* Now, we are left to proving

$$\Delta_n^{-1/2}\Big(V(f)_t^n - V(f)_t\Big) - L(f)_t^n \xrightarrow{\text{u.c.p.}} \int_0^t b_s ds,$$

where the process $(b_s)_{s\geq 0}$ is given in Theorem 3.2. In view of the previous step, it is sufficient to show that

$$\Delta_n^{-1/2} \sum_{i=1}^{[t/\Delta_n]} \int_{(i-1)\Delta_n}^{i\Delta_n} (\rho_{\sigma_s}(f) - \rho_{\sigma_{(i-1)\Delta_n}}(f))ds \xrightarrow{\text{u.c.p.}} 0, \tag{3.13}$$

$$\Delta_n^{1/2} \sum_{i=1}^{[t/\Delta_n]} \mathbb{E}\Big[f\Big(\frac{\Delta_i^n X}{\sqrt{\Delta_n}}\Big) - f(\alpha_i^n)|\mathcal{F}_{(i-1)\Delta_n}\Big] \xrightarrow{\text{u.c.p.}} \int_0^t b_s ds. \tag{3.14}$$

To sketch the following ideas we use the notation $Y^n \asymp X^n$ whenever $Y^n - X^n \stackrel{\text{u.c.p.}}{\longrightarrow} 0$. We start with the convergence in (3.13). Notice that the function $\rho_{\cdot}(f) : \mathbb{R} \to \mathbb{R}$ is differentiable. Hence, we have

$$\begin{aligned}\rho_{\sigma_s}(f) - \rho_{\sigma_{(i-1)\Delta_n}}(f) &\approx \rho'_{\sigma_{(i-1)\Delta_n}}(f)(\sigma_s - \sigma_{(i-1)\Delta_n}) \\ &\approx \rho'_{\sigma_{(i-1)\Delta_n}}(f)\Big(\tilde{\sigma}_{(i-1)\Delta_n}(W_s - W_{(i-1)\Delta_n}) + \tilde{\tau}_{(i-1)\Delta_n}(V_s - V_{(i-1)\Delta_n})\Big) \\ &:= \chi_i^n(s)\end{aligned}$$

due to condition condition (3.9). Notice that

$$\begin{aligned}Var\Big[\Delta_n^{-1/2} \sum_{i=1}^{[t/\Delta_n]} \int_{(i-1)\Delta_n}^{i\Delta_n} \chi_i^n(s)\, ds\Big] &= \Delta_n^{-1} \sum_{i=1}^{[t/\Delta_n]} \mathbb{E}\Big[\Big(\int_{(i-1)\Delta_n}^{i\Delta_n} \chi_i^n(s)\, ds\Big)^2\Big] \\ &= C\Delta_n \to 0\end{aligned}$$

Hence, we deduce (3.13).
Finally, let us highlight the proof of (3.14). The most important idea is the following approximation step

$$\begin{aligned}&\Delta_n^{1/2} \sum_{i=1}^{[t/\Delta_n]} \mathbb{E}\Big[f\Big(\frac{\Delta_i^n X}{\sqrt{\Delta_n}}\Big) - f(\alpha_i^n)|\mathcal{F}_{(i-1)\Delta_n}\Big] \\ &\asymp \Delta_n^{1/2} \sum_{i=1}^{[t/\Delta_n]} \mathbb{E}\Big[f'(\alpha_i^n)\Big(\frac{\Delta_i^n X}{\sqrt{\Delta_n}} - \alpha_i^n\Big)|\mathcal{F}_{(i-1)\Delta_n}\Big] \\ &\asymp \sum_{i=1}^{[t/\Delta_n]} \mathbb{E}\Big[f'(\alpha_i^n)\Big(\Delta_n a_{(i-1)\Delta_n} + \int_{(i-1)\Delta_n}^{i\Delta_n} (\sigma_s - \sigma_{(i-1)\Delta_n})dW_s\Big)|\mathcal{F}_{(i-1)\Delta_n}\Big] \\ &\asymp \sum_{i=1}^{[t/\Delta_n]} \mathbb{E}\Big[f'(\alpha_i^n)\Big(\Delta_n a_{(i-1)\Delta_n} + \tilde{\sigma}_{(i-1)\Delta_n} \underbrace{\int_{(i-1)\Delta_n}^{i\Delta_n} (W_s - W_{(i-1)\Delta_n})dW_s}_{=\frac{1}{2}(|\Delta_i^n W|^2 - \Delta_n) \text{ due to Itô's formula}}\Big)|\mathcal{F}_{(i-1)\Delta_n}\Big]. \\ &\asymp \Delta_n \sum_{i=1}^{[t/\Delta_n]} \Big[a_{(i-1)\Delta_n}\rho_{\sigma_{(i-1)\Delta_n}}(f') + \frac{1}{2}\tilde{\sigma}_{(i-1)\Delta_n}\big\{\rho_{\sigma_{(i-1)\Delta_n}}(f', 2) - \rho_{\sigma_{(i-1)\Delta_n}}(f')\big\}\Big] \\ &\stackrel{\text{u.c.p.}}{\longrightarrow} \int_0^t b_s ds,\end{aligned}$$

which completes the proof of Theorem 3.2. □

4 Robust Estimation

In this section we consider semimartingales with jumps of the form

$$X_t = X_0 + \int_0^t a_s\,ds + \int_0^t \sigma_s\,dW_s + \sum_{j=1}^{N_t} Y_j, \tag{4.1}$$

where the processes $(a_s)_{s\geq 0}$, $(\sigma_s)_{s\geq 0}$ satisfy the same assumptions as in section 3, $(N_s)_{s\geq 0}$ denotes a Poisson process and $(Y_j)_{j\geq 0}$ is a sequence of iid random variables. Sometimes we denote the jump part of X by X^j, i.e.

$$X_t^j = \sum_{k=1}^{N_t} Y_k \tag{4.2}$$

Processes of the form (4.2) are called "compound Poisson processes"; they exhibit finitely many jumps on finite intervals.
In this subsection we are interested in constructing estimators of integrated (powers of) volatility, which are robust to the presence of jumps. This is particularly important for a separate estimation of

$$[X^c]_t = \int_0^t \sigma_s^2\,ds, \qquad [X^j]_t = \sum_{0\leq s\leq t} |\Delta X_s|^2 = \sum_{k=1}^{N_t} Y_k^2.$$

In finance these two parts of the quadratic variation have a different interpretation. We will present two methods of robust estimation: (a) Multipower variation, which goes back to Barndorff-Nielsen and Shephard (see [4]), (b) Threshold estimation proposed by Mancini in [14]. There exist other alternative methods, see e.g. [7].

4.1 Multipower Variation

Multipower variation is a straightforward extension of the power variation concept. It is defined as

$$V(X, p_1, \ldots, p_k)_t^n = \Delta_n^{1-p^+/2} \sum_{i=1}^{\lfloor t/\Delta_n \rfloor - k+1} |\Delta_i^n X|^{p_1} \ldots |\Delta_{i+k-1}^n X|^{p_k}, \tag{4.3}$$

where $p_j \geq 0$ and $p^+ = p_1 + \ldots + p_k$. As we will see in the next theorem, the asymptotic behaviour of multipower variation is very similar to the asymptotic behaviour of power variation, if X is continuous.

Theorem 4.1
Assume that X is a continuous semimartingale, i.e. $X^j = 0$.

(i) For any $p_1, \dots, p_k \geq 0$, it holds that

$$V(X, p_1, \dots, p_k)_t^n \xrightarrow{u.c.p.} V(X, p_1, \dots, p_k)_t := m_{p_1} \dots m_{p_k} \int_0^t |\sigma_s|^{p^+} \, ds, \tag{4.4}$$

with $m_p = \mathbb{E}[|N(0,1)|^p]$.

(ii) Assume that the conditions of Theorem 3.2 hold. Then we obtain

$$\Delta_n^{-1/2}\Big(V(X, p_1, \dots, p_k)_t^n - V(X, p_1, \dots, p_k)_t\Big) \xrightarrow{st} L(p_1, \dots, p_k)_t = \sqrt{A_{p_1,\dots,p_k}} \int_0^t |\sigma_s|^{p^+} \, dW_s', \tag{4.5}$$

where the constant $A_{p_1,\dots,p_k}$ is defined by

$$A_{p_1,\dots,p_k} = \prod_{l=1}^k m_{2p_l} - (2k-1) \prod_{l=1}^k m_{2p_l}^2 + 2 \sum_{l=1}^{k-1} \prod_{j=1}^l m_{p_j} \prod_{j=k-l+1}^k m_{p_j} \prod_{j=1}^{k-l} m_{p_j + p_{j+l}}$$

Remark 4.1
Notice that the class of multipower variations estimates the same objects as in Example 3.1. So, it is a priori not clear why these statistics can be more useful. However, they have a more rich behaviour when jumps are present, as we will see in the next theorem. Theorem 4.1 has been proved in [3]. The proof mainly follows the same ideas as presented in section 3.

Now, we turn our attention to the asymptotic behaviour of $V(X, p_1, \dots, p_k)_t^n$ in the presence of jumps.

Theorem 4.2
Assume that X satisfies the representation (4.1).

(i) For any $p_1, \dots, p_k \geq 0$ with $\max_j p_j < 2$, it holds that

$$V(X, p_1, \dots, p_k)_t^n \xrightarrow{u.c.p.} V(X^c, p_1, \dots, p_k)_t = m_{p_1} \dots m_{p_k} \int_0^t |\sigma_s|^{p^+} \, ds. \tag{4.6}$$

(ii) Assume that the conditions of Theorem 3.2 hold. For any $p_1, \dots, p_k \geq 0$ with $\max_j p_j < 1$ we obtain:

$$\Delta_n^{-1/2}\Big(V(X, p_1, \dots, p_k)_t^n - V(X, p_1, \dots, p_k)_t\Big) \xrightarrow{st} L(p_1, \dots, p_k)_t, \tag{4.7}$$

where the limit $L(p_1, \dots, p_k)_t$ was defined in Theorem 4.1.

In other words, Theorem 4.2 makes it possible to estimate integrated volatility robustly to jumps. The most important example is the following:

$$V(X,1,1)_t^n = \sum_{i=1}^{\lfloor t/\Delta_n \rfloor - 1} |\Delta_i^n X||\Delta_{i+1}^n X| \xrightarrow{\text{u.c.p.}} m_1^2 \int_0^t \sigma_s^2 \, ds. \tag{4.8}$$

This result shold be compared to the well known convergence

$$V(X,2)_t^n = \sum_{i=1}^{\lfloor t/\Delta_n \rfloor} |\Delta_i^n X|^2 \xrightarrow{\text{u.c.p.}} \int_0^t \sigma_s^2 \, ds + \sum_{0\le s\le t} |\Delta X_s|^2. \tag{4.9}$$

Combining these two results, we can construct estimators for $[X^c]_t = \int_0^t \sigma_s^2 \, ds$ and $[X^j]_t = \sum_{0\le s\le t} |\Delta X_s|^2$. We will use it later to propose a test for jumps (see [5]).

Remark 4.2
For a general jump process X^j we need to impose stronger assumptions for Theorem 4.2 (ii). In particular, the jump part X^j of X must have finite variation. □

A test for jumps:
The convergence in (4.8), (4.9) suggests that we may use the statistic

$$V(X,2)_t^n - m_1^{-2} V(X,1,1)_t^n \xrightarrow{\mathbb{P}} \sum_{0\le s\le t} |\Delta X_s|^2$$

to test whether X has jumps or not. Under the null hypothesis of no jumps, one can show that

$$\Delta_n^{-1/2} \Big(V(X,2)_t^n - m_1^{-2} V(X,1,1)_t^n \Big) \xrightarrow{st} MN\Big(0, \mu \int_0^t \sigma_s^4 \, ds\Big)$$

with $\mu = \frac{\pi^2}{4} + \pi - 5$. Theorem 4.2 (i) implies that

$$V(X,1,1,1,1)_t^n \xrightarrow{\text{u.c.p.}} m_1^4 \int_0^t \sigma_s^4 \, ds.$$

The test statistic for jumps is defined as

$$S_t^n = \frac{\Delta_n^{-1/2} \Big(V(X,2)_t^n - m_1^{-2} V(X,1,1)_t^n \Big)}{\sqrt{\mu \, m_1^{-4} V(X,1,1,1,1)_t^n}}.$$

The null hypothesis of no jumps is rejected when

$$S_t^n > C_{1-\alpha}$$

where $C_{1-\alpha}$ is the $(1-\alpha)$-quantile of $N(0,1)$. Indeed it holds that

$$\mathbb{P}_{H_0}(S_t^n > C_{1-\alpha}) \xrightarrow{n\to\infty} \alpha$$
$$\mathbb{P}_{H_1}(S_t^n > C_{1-\alpha}) \xrightarrow{n\to\infty} 1$$

□

Sketch of the proof of Theorem 4.2:
First, let us set

$$V(X,p_1,\ldots,p_k)_t^n = \sum_{i=1}^{\lfloor t/\Delta_n\rfloor-k+1} \chi_i^n, \qquad \chi_i^n = \Delta_n^{1-p^+/2}|\Delta_i^n X|^{p_1}\ldots|\Delta_{i+k-1}^n X|^{p_k}.$$

The proof is performed through the following observations:

(i) Recall that $X_t^j = \sum_{l=1}^{N_t} Y_l$ and $(N_s)_{s\geq 0}$ has only finitely many jumps on the interval $[0,t]$. This means that only finitely many χ_i^n's are influenced by X^j. □

(ii) Supposed that some χ_i^n is influenced by the jump part X^j. Then, with probability converging to 1, only one of the intervals $[(i-1)\Delta_n, i\Delta_n]\ldots[(i+k-2)\Delta_n,(i+k-1)\Delta_n]$ contains a jump of N. Indeed, it is easily shown that

$$\mathbb{P}\big(N \text{ has } 2 \text{ or more jumps in the intervals } [(i-1)\Delta_n, i\Delta_n]\ldots[(i+k-2)\Delta_n,(i+k-1)\Delta_n]\big)$$
$$= O_{\mathbb{P}}(\Delta_n^2).$$

□

(iii) For the continuous part X^c we have the approximation

$$\Delta_i^n X^c = O_{\mathbb{P}}(\Delta_n^{1/2}).$$

□

(iv) Putting things together we obtain the following: Let χ_i^n be a summand influenced by the jump part X^j (there are only finitely many of these!). W.l.o.g. this jump lies in the interval $[(i-1)\Delta_n, i\Delta_n]$. Then:

$$\chi_i^n = \Delta_n^{1-p^+/2}|\Delta_i^n X|^{p_1}\underbrace{|\Delta_{i+1}^n X|^{p_2}}_{=O_{\mathbb{P}}(\Delta_n^{p_2/2})}\ldots\underbrace{|\Delta_{i+k-1}^n X|^{p_k}}_{=O_{\mathbb{P}}(\Delta_n^{p_k/2})} = O_{\mathbb{P}}\big(\Delta_n^{1-p_1/2}\big).$$

Now, it is immediately clear that we deduce Theorem 4.2 (i) if $\max_j p_j < 2$ and Theorem 4.2 (ii) if $\max_j p_j < 1$. □

4.2 Treshold Estimation

In this subsection we present a different approach of robust estimation proposed by Mancini (see [14]). We restrict our attention to robust estimation of the integrated volatility $[X^c]_t = \int_0^t \sigma_s^2\, ds$. The treshold estimator is defined as

$$TRV_t^n := \sum_{i=1}^{\lfloor t/\Delta_n\rfloor} |\Delta_i^n X|^2 \mathbf{1}_{\{|\Delta_i^n X|\leq C\Delta_n^{\varpi}\}}, \tag{4.10}$$

where the discontinuous semimartingale is given by (4.1), $C > 0$ and $\omega \in (0, \frac{1}{2})$. The idea behind this estimator is relatively simple. The increments of the continuous part X^c satisfy

$$\Delta_i^n X^c = O_{\mathbb{P}}\big(\Delta_n^{1/2}\big),$$

so with probability converging to 1 they fulfill $|\Delta_i^n X| \leq C\Delta_n^{\omega}$ for $\omega < \frac{1}{2}$. On the other hand, if there is a jump in the interval $[(i-1)\Delta_n, i\Delta_n]$ (which happens only for finitely many i's in our model), then $|\Delta_i^n X| > C\Delta_n^{\omega}$ with probability converging to 1, since $\Delta_n^{\omega} \to 0$. The following result is then rather sraightforward.

Theorem 4.3
Assume that the process X satisfies (4.1). *Then it holds that*

$$TRV_t^n \xrightarrow{\text{u.c.p.}} [X^c]_t = \int_0^t \sigma_s^2 \, ds, \tag{4.11}$$

and

$$\Delta_n^{-1/2}\Big(TRV_t^n - \int_0^t \sigma_s^2 \, ds\Big) \xrightarrow{st} \sqrt{2}\int_0^t \sigma_s^2 \, dW_s'. \tag{4.12}$$

Remark 4.3
Observing Examples 3.1 and 3.2 we conclude that the treshold estimator TRV_t^n is robust to the presence of (finite activity) jumps. For more general jump processes we require some stronger conditions to prove (4.12). *In particular, the jump part X^j must have finite variation.* □

Using the result of Theorem 4.3 we can estimate $[X^c]_t$ and $[X^j]_t$ separately:

$$TRV_t^n \xrightarrow{\text{u.c.p.}} \int_0^t \sigma_s^2 \, ds,$$

$$\sum_{i=1}^{\lfloor t/\Delta_n \rfloor} |\Delta_i^n X|^2 - TRV_t^n = \sum_{i=1}^{\lfloor t/\Delta_n \rfloor} |\Delta_i^n X|^2 \mathbf{1}_{\{|\Delta_i^n X| > C\Delta_n^{\omega}\}} \xrightarrow{\mathbb{P}} \sum_{0\leq s\leq t} |\Delta X_s|^2.$$

5 Limit Theorems for Discontinuous Semimartingales

Until now we have learned how to estimate volatility in continuous models and how to construct robust estimators in discontinuous models. In this section we concentrate on central limit theorems for realised jumps, e.g. for estimators of

$$[X]_t = \int_0^t \sigma_s^2\, ds + \sum_{0\le s\le t} |\Delta X_s|^2.$$

We consider again the discontinuous model

$$X_t = X_0 + \int_0^t a_s\, ds + \int_0^t \sigma_s\, dW_s + \sum_{j=1}^{N_t} Y_j, \tag{5.1}$$

as in section 4. The statistics of interest are again power variations

$$\overline{V}(f)_t^n = \sum_{i=1}^{[t/\Delta_n]} f(\Delta_i^n X), \qquad f(x) = |x|^p. \tag{5.2}$$

The first result is the law of large numbers, which is due to [13].

Theorem 5.1
Assume that the process X is of the form (5.1) *and $f(x) = |x|^p$ for $p > 0$. For any $t \ge 0$ we have*

$$\overline{V}(f)_t^n \xrightarrow{\mathbb{P}} \overline{V}(f)_t = \begin{cases} \sum\limits_{0\le s\le t} |\Delta X_s|^p, & p > 2, \\ [X]_t = \int_0^t \sigma_s^2 ds + \sum\limits_{0\le s\le t} |\Delta X_s|^2, & p = 2. \end{cases}$$

Remark 5.1
The convergence for $p = 2$ is, of course, a well-known result. For $p > 2$, the result is easily shown (for the model (5.1)*!) via a combination of*

$$\sum_{i=1}^{[t/\Delta_n]} |\Delta_i^n X^c|^p \xrightarrow{\mathbb{P}} 0 \quad \textit{and} \quad \sum_{i=1}^{[t/\Delta_n]} |\Delta_i^n X^j|^p \xrightarrow{\mathbb{P}} \sum_{0\le s\le t} |\Delta X_s|^2 \quad \forall\, p > 2.$$

To proceed with the associated central limit theorem we need to introduce some notation. On an extension $(\Omega', \mathbb{F}', \mathbb{P}')$ of the original probability space $(\Omega, \mathbb{F}, \mathbb{P})$, let $(U_m)_{m\in\mathbb{N}}$ and $(U'_m)_{m\in\mathbb{N}}$ be iid $N(0,1)$-distributed sequences, $(\kappa_m)_{m\in\mathbb{N}}$ be an iid $U([0,1])$-distributed sequence, and $(W'_t)_{t\geq 0}$ be a Brownian motion. All these processes are mutually independent and independent of $\mathbb{F}$. We define

$$\overline{L}(f)_t = \sum_{m:\ T_m\leq t} f'(\Delta X_{T_m})\Big(\sqrt{\kappa_m}U_m\sigma_{T_m-} + \sqrt{1-\kappa_m}U'_m\sigma_{T_m}\Big), \tag{5.3}$$

where $(T_m)_{m\geq 1}$ are jump times of N (and so of X) and $f(x) = |x|^p$. Jacod (see [11]) has proved the following result.

Theorem 5.2
Let $f(x) = |x|^p$ for $p > 0$. For any $t \geq 0$ we obtain

$$\Delta_n^{-1/2}\Big(\overline{V}(f)_t^n - \overline{V}(f)_t\Big) \xrightarrow{st} \begin{cases} \overline{L}(f)_t & p > 3, \\ \overline{L}(f)_t + L(f)_t, & p = 2. \end{cases} \tag{5.4}$$

where $\overline{L}(f)_t$ is defined by (5.3) *and $L(f)_t$ is given via*

$$L(f)_t = \sqrt{2}\int_0^t \sigma_s^2\, dW'_s.$$

We remark that the result of Theorem 5.2 does not hold in a functional sense. A functional central limit theorem can be obtained by replacing $V(f)_t$ through $V(f)_{\Delta_n\lfloor t/\Delta_n\rfloor}$. Notice that there is no central limit theorem for $p \in (2,3]$.

Remark 5.2 (Statistical applications)
The second part of Theorem 5.2 implies that

$$\Delta_n^{-1/2}\Big(\sum_{i=1}^{\lfloor t/\Delta_n\rfloor}|\Delta_i^n X|^2 - [X]_t\Big) \xrightarrow{st} \overline{L}(f)_t + L(f)_t$$

with $f(x) = x^2$. How can we use this result to construct confidence regions for $[X]_t$? Assume that the processes X and σ have no common jumps. Then it holds that

$$\overline{L}(f)_t = 2\sum_{m:\ T_m\leq t}\Delta X_{T_m}\sigma_{T_m}\Big(\underbrace{\sqrt{\kappa_m}U_m + \sqrt{1-\kappa_m}U'_m}_{\sim N(0,1)}\Big).$$

Consequently, we have that

$$\overline{L}(f)_t + L(f)_t \sim MN\Big(0, 2\int_0^t \sigma_s^4\, ds + 4\sum_{m:\ T_m\leq t}\sigma_{T_m}^2|\Delta X_{T_m}|^2\Big).$$

The first part of the conditional variance

$$V^2 = \int_0^t \sigma_s^2\, ds + 4\sum_{m:\ T_m\leq t}\sigma_{T_m}^2|\Delta X_{T_m}|^2$$

can be estimated as in example 3.2; for the estimation of the second part we need local estimates of the volatility. Since σ is càdlàg, the local estimate of σ_s^2 is given as

$$\sigma_{s,n}^2 := \frac{1}{\kappa_n \Delta_n} \sum_{i=\lfloor s/\Delta_n \rfloor}^{\lfloor s/\Delta_n \rfloor + \kappa_n} |\Delta_i^n X|^2 \mathbf{1}_{\{|\Delta_i^n X| \leq C\Delta_n^\omega\}}$$

with $C > 0$, $\omega \in (0, \frac{1}{2})$ and $\kappa_n \to \infty$, $\Delta_n \kappa_n \to 0$. Indeed, the results of section 4.2 imply that

$$\sigma_{s,n}^2 \xrightarrow{\mathbb{P}} \sigma_s^2.$$

Now, the estimator of V^2 can be defined as

$$V_n^2 = \frac{2}{3\Delta_n} \sum_{i=1}^{\lfloor t/\Delta_n \rfloor} |\Delta_i^n X|^4 + 4 \sum_{i=1}^{\lfloor t/\Delta_n \rfloor} \sigma_{i\Delta_n,n}^2 |\Delta_i^n X|^2 \mathbf{1}_{\{|\Delta_i^n X| > C\Delta_n^\omega\}} \xrightarrow{\mathbb{P}} V^2.$$

Finally, we obtain a standard central limit theorem

$$\frac{\Delta_n^{-1/2}\Big(\sum_{i=1}^{\lfloor t/\Delta_n \rfloor} |\Delta_i^n X|^2 - [X]_t\Big)}{V_n} \xrightarrow{d} N(0,1)$$

for any fixed $t > 0$. □

Remark 5.3 (Another test for jumps)
Ait-Sahalia and Jacod (see [1]) applied the result of Theorem 5.2 to construct a test for jumps. The main idea is to use ratio statistics at two different frequencies. Namely, Theorem 3.2 implies the convergence

$$\frac{\sum_{i=1}^{\lfloor t/\Delta_n \rfloor} |\Delta_i^n X|^4}{\sum_{i=2}^{\lfloor t/\Delta_n \rfloor} \Big(X_{i\Delta_n} - X_{(i-2)\Delta_n}\Big)^4} \xrightarrow{\mathbb{P}} \begin{cases} 1: & \text{if } X \text{ has jumps}, \\ \frac{1}{2}: & \text{if } X \text{ has no jumps}. \end{cases}$$

Furthermore, we have the associated central limit theorems in both cases. This gives a possibility to construct a consistent level-α test for the null hypothesis of no jumps. □

Sketch of the proof of theorem 5.2:
We start with the case $p > 3$, $f(x) = |x|^p$. Observe the decomposition

$$\Delta_n^{-1/2}\Big(\overline{V}(f)_t^n - \overline{V}(f)_t\Big) = S_t^n + R_t^n,$$

with

$$S_t^n = \Delta_n^{-1/2} \sum_{i=1}^{\lfloor t/\Delta_n \rfloor} \Big\{ f(\Delta_i^n X) - f(\Delta_i^n X^j) \Big\},$$

$$R_t^n = \Delta_n^{-1/2} \Big(\sum_{i=1}^{\lfloor t/\Delta_n \rfloor} f(\Delta_i^n X^j) - \sum_{0 \leq s \leq t} f(\Delta X_s) \Big).$$

First of all, since the probability of having two or more jumps in the interval $[(i-1)\Delta_n, i\Delta_n]$ is very small, we have

$$R_t^n \xrightarrow{\mathbb{P}} 0.$$

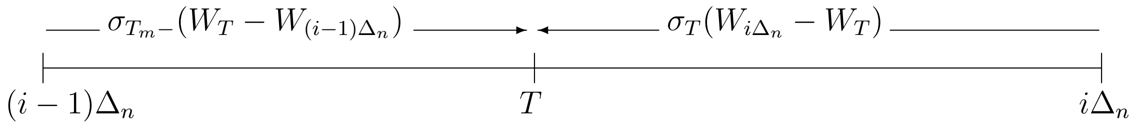
For the term S_t^n we obtain that

$$\begin{aligned} S_t^n &\sim \Delta_n^{-1/2} \sum_{i=1}^{\lfloor t/\Delta_n \rfloor} f'(\Delta_i^n X^j)\big(\Delta_i^n X - \Delta_i^n X^j\big) \\ &\sim \Delta_n^{-1/2} \sum_{i=1}^{\lfloor t/\Delta_n \rfloor} f'(\Delta_i^n X^j) \int_{(i-1)\Delta_n}^{i\Delta_n} \sigma_s \, dW_s. \end{aligned}$$

Now we proceed with a rather intuitive argument. Assume that X has one jump at time $T \in [(i-1)\Delta_n, i\Delta_n]$ (and with probability converging to 1 it is the only one in this interval). Then

$$\Delta_n^{-1/2} f'(\Delta_i^n X^j) \int_{(i-1)\Delta_n}^{i\Delta_n} \sigma_s \, dW_s \;\sim\; \Delta_n^{-1/2} f'(\Delta X_T)\Big[\int_{(i-1)\Delta_n}^{T} \sigma_s \, dW_s + \int_{T}^{i\Delta_n} \sigma_s \, dW_s \Big]$$

Since σ is càdlàg, the above quantity is approximated as illustrated below:



Because N is a Poisson process, T is uniformly distributed in $[(i-1)\Delta_n, i\Delta_n]$. This implies that

$$\Delta_n^{-1/2} f'(\Delta X_T)\Big[\int_{(i-1)\Delta_n}^{T} \sigma_s \, dW_s + \int_{T}^{i\Delta_n} \sigma_s \, dW_s \Big] \xrightarrow{st} f'(\Delta X_T)\Big[\sqrt{\kappa}\, U \, \sigma_t + \sqrt{1-\kappa}\, U' \, \sigma_T \Big],$$

with $\kappa \sim U([0,1])$, $U, U' \sim N(0,1)$ and all these variables are independent (and also independent of $\mathbb{F}$). Summing up over all jump times we obtain

$$\Delta_n^{-1/2}\Big(\overline{V}(f)_t^n - \overline{V}(f)_t \Big) \xrightarrow{st} \overline{L}(f)_t$$

for $f(x) = |x|^p$ for $p > 3$. □

For the case $f(x) = |x|^2$ the situation is simple:

$$\Delta_n^{-1/2}\Big(\sum_{i=1}^{\lfloor t/\Delta_n \rfloor} |\Delta_i^n X|^2 - [X]_t \Big) = S_t^n + \overline{S}_t^n + R_t^n,$$

with

$$S_t^n = \Delta_n^{-1/2}\Big(\sum_{i=1}^{\lfloor t/\Delta_n\rfloor} |\Delta_i^n X^c|^2 - \int_0^t \sigma_s^2\,ds\Big) \xrightarrow{st} L(f)_t \qquad \textit{(Example 3.2)},$$

$$\overline{S}_t^n = \Delta_n^{-1/2} \sum_{i=1}^{\lfloor t/\Delta_n\rfloor} \Delta_i^n X^c \Delta_i^n X^j \xrightarrow{st} \overline{L}(f)_t \qquad \textit{(as before)},$$

$$R_t^n = \Delta_n^{-1/2}\Big(\sum_{i=1}^{\lfloor t/\Delta_n\rfloor} |\Delta_i^n X^j|^2 - \sum_{0\le s\le t} |\Delta X_s|^2\Big) \xrightarrow{\mathbb{P}} 0 \qquad \textit{(as before)}.$$

□

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