Mean Field Bolker-Pacala Models of Population Dynamics

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April, 2016

Outline

- Description of mean field model for one population
  - Model and random walk
  - Limit theorems
- Extension of mean field model to multiple, e.g., stratified, populations
  - Multiple equilibria
  - Global limit theorems and ergodicity

Basic model: Three processes

- Infinite initial population of particles living on the lattice: $n(0, x),\ x \in \mathbb{Z}^d$
- 1. Birth and migration
  - Each particle, in time *dt*, produces 1 offspring with probability *b dt*
  - Offspring migrates distance *z* on lattice with probability $a^+(z)$

Basic model: Three processes

- 2. Mortality
  - Each particle, in time *dt*, dies with probability $\mu\, dt$
- 3. Competition
  - For any two particles, located at *x* and *y*, in time *dt*, probability that each dies is $\alpha^{-}(x,y)\, dt$
  - May assume that both do not die
  - Probability that particle dies due to competition is sum due to competition with all other particles

Mean field approximation

- Consider box $Q_L \subset \mathbb{Z}^d$ with $|Q_L| = L$, *L* a large parameter.
- Let $a^+(x) = \frac{\kappa}{L}$ on Q_L, $\quad a^-(x) = \frac{\gamma}{L^2}$ on Q_L $\quad \gamma > 0$

- Total number of particles $N_L(t) = \sum_{x \in Q_L} n(t,x)$ is the logistic Markov chain
- Transition rates

$$P\left(N_L(t+dt) = j | N_L(t) = n\right) = \begin{cases} nb\,dt + o(dt^2) & \text{if } j = n+1 \\ n\mu\,dt + \gamma n^2/L\,dt + o(dt^2) & \text{if } j = n-1 \\ o(dt^2) & \text{otherwise} \end{cases}$$

Mean field approximation

- Modified l.c. to eliminate absorption at 0:

  Generator

  $$\mathcal{L}\psi(n) = \alpha_n \psi(n-1) - (\alpha_n + \beta_n)\psi(n) + \beta_n \psi(n+1)$$, *n*>0

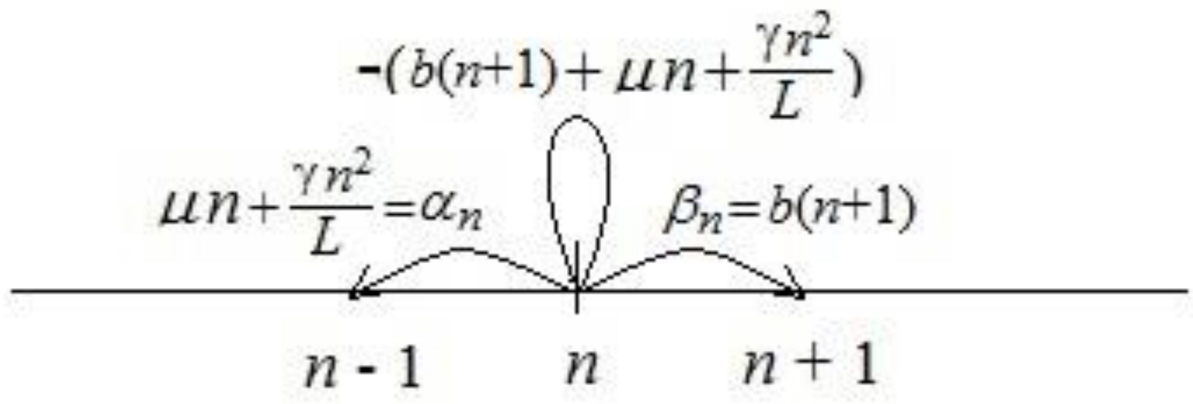
  $$\mathcal{L}\psi(0) = \beta_0 \psi(1) - \beta_0 \psi(0)$$

  Transition rates

  $$\beta_n = b(n+1), n \geq 0 \quad \alpha_n = \mu n + \frac{\gamma n^2}{L}, n \geq 1$$

Mean field approximation

As random walk



– Equilibrium for appropriately chosen large *L*

$$\tilde{n}_L^* = \frac{(b-\mu)L}{\gamma}$$

– Asymptotics same as for logistic chain

Mean field approximation

- Local Central Limit Theorem
  - Let $b > \mu$. If $k = O(L^{2/3})$ then

    for the invariant distribution π_L, where $\sigma_L^2 = Lb/\gamma$

$$\pi_L(n_L^* + k) \sim \frac{e^{-k^2/2\sigma_L^2}}{\sqrt{2\pi\sigma_L^2}}$$

Mean field approximation

- Proof uses well-known formula for invariant distribution of birth-and-death process

$$\pi(x) = \lim_{t\to\infty} p(t,\cdot,x) = \begin{cases} S^{-1}, & x = 0 \\ S^{-1}\frac{\beta_0\ldots\beta_{x-1}}{\alpha_1\ldots\alpha_x}, & x > 0 \end{cases}$$

Where

$$S = 1 + \frac{\beta_0}{\alpha_1} + \frac{\beta_0\beta_1}{\alpha_1\alpha_2} + \ldots + \frac{\beta_0\ldots\beta_n}{\alpha_1\ldots\alpha_{n+1}} + \ldots.$$

And fact that for logistic chain, *S* can be represented in terms of degenerated hypergeometric function.

Mean field approximation

- Global Central Limit Theorem (Kurtz-type result)
  - If $b > \mu$, $L > 0$, $\gamma > 0$ then

$$\frac{N_L(t) - \frac{(b-\mu)L}{\gamma}}{\sqrt{L}} \xrightarrow[L\to\infty]{\text{law}} \zeta(t)$$

where $\zeta(t)$ is an Ornstein-Uhlenbeck process.

[1] M. Bessonov, S. Molchanov. and J. Whitmeyer, *A Mean Field Approximation of the Bolker-Pacala Population Model.* Markov Processes and Related Fields, 20, (2014) 329-348.

Multi-Class Extension: *N*-Box Model

- Instead of 1-box model, *N*-box model
- Gives rise to a random walk on

$$(\mathbb{Z}_+)^N = \{(n_1, n_2, \ldots, n_N) : n_i \in \mathbb{Z}_+, 1 \le i \le N\}$$

- Migration potential:

$$a_L^+(x,y) = a_{ij}^+/L, \quad i,j = 1,2,\ldots,N$$

- Competition potential:

$$a_L^-(x,y) = a_{ij}^-/L^2, \quad i,j = 1,2,\ldots,N$$

- Population given by

$$n(t) = \{n_1(t), n_2(t), \ldots, n_N(t)\}$$

N-Box Model

- Change in time dt:

$$n(t+dt|n(t)) = n(t) + \begin{cases} e_i & \text{w. pr. } bn_i(t)dt + o(dt^2) \\ -e_i & \text{w. pr. } \mu n_i(t)dt + \dfrac{n_i(t)}{L}\displaystyle\sum_{j=1}^{N} a_{ij}^{-} n_j(t)dt + o(dt^2) \\ e_j - e_i & \text{w. pr. } n_i(t)a_{ij}^{+}dt + o(dt^2), \quad j \neq i \\ 0 & \text{w. pr. } 1 - \displaystyle\sum_{i=1}^{N}(b_i + \mu_i)n_i(t)dt \\ & \qquad -\dfrac{1}{L}\displaystyle\sum_{i,j} n_i(t)n_j(t)a_{ij}^{-}dt + \sum_{i,j} n_i(t)a_{ij}^{+} + o(dt^2) \\ \text{other} & \text{w. pr. } o(dt^2) \end{cases}$$

Where e_i is vector with 1 in position i and 0 everywhere else.

Functional LLN and CLT for *N*-Box Model

- *Transition function p* from probabilities above:

$$p((\mathbf{n}(t), \mathbf{n}(t)+\mathbf{l}) = \begin{cases} bn_i(t) & \mathbf{l} = e_i \\ \mu n_i(t) + \dfrac{n_i(t)}{L} \displaystyle\sum_{j=1}^{N} a_{ij}^{-} n_j(t) & \mathbf{l} = -e_i \\ n_i(t) a_{ij}^{+} & \mathbf{l} = e_j - e_i,\ j \neq i \\ -\displaystyle\sum_{i=1}^{N} (b_i + \mu_i) n_i(t) - \dfrac{1}{L} \displaystyle\sum_{i,j} n_i(t) n_j(t) a_{ij}^{-} + & \mathbf{l} = 0 \\ \qquad + \displaystyle\sum_{i,j} n_i(t) a_{ij}^{+} & \\ 0 & \text{all other } \mathbf{l} \end{cases}$$

Functional LLN and CLT for *N*-Box Model

- Rescale process. Temporarily fix *L*.

- Set, for all *i*, $z_i(t) := \dfrac{n_i(t)}{L}$

- Define $f_L(\mathbf{z}(t), \mathbf{l}) := \dfrac{1}{L} p(\mathbf{n}(t), \mathbf{n}(t) + \mathbf{l})$

Functional LLN and CLT for N-Box Model

- Then

$$f_L(\mathbf{z}(t),\mathbf{l}) = \begin{cases} b_i z_i & \mathbf{l} = e_i, \quad i = 1,\dots,N \\ \mu_i z_i + a_{i,i}^- z_i^2 + \sum_{j\neq i} a_{i,j}^- z_i z_j & \mathbf{l} = -e_i, \quad i = 1,\dots,N \\ a_{i,j}^+ z_i & \mathbf{l} = e_j - e_i, \quad i,j = 1,\dots,N; i \neq j \\ (\text{not needed}) & \mathbf{l} = \mathbf{0} \\ 0 & \text{otherwise} \end{cases}$$

- Note that f_L does not depend on L.
- Set $F_i(\mathbf{z}(t)) := \sum_{l_i=-1}^{1} l_i f(\mathbf{z}(t),\cdot)$
- Now letting L vary, relabel $z_{Li}(t) := \dfrac{n_i(t)}{L}$,
  and $Z_L(t) = (z_{L1}(t),\dots,z_{LN}(t))$

Functional LLN and CLT for *N*-Box Model

For the rescaled system we have a functional Law of Large Numbers, following papers by Kurtz.

Theorem. The process $Z_L(t)$ converges uniformly in probability as $L \rightarrow \infty$ to a deterministic process, the solution of the system of differential equations

$$\frac{d\mathbf{z}(t)}{dt} = \mathbf{F}(\mathbf{z}(t))$$

with initial point a stable equilibrium z^* of the system, i.e., solution of $\mathbf{0} = \mathbf{F}(\mathbf{z}(t))$

Functional LLN and CLT for *N*-Box Model

Similarly, we have a functional Central Limit Theorem.

In particular, $\zeta_L(t) := \sqrt{L}(Z_L(t) - z^*)$ converges weakly, as $L \to \infty$, to an Ornstein-Uhlenbeck process

Results for N = 2 and 3 Boxes

- Assume completely symmetric conditions
  - Single birth rate β, single death rate μ
  - Equal internal competition or “suppression” rates: $a_I^- = a_{ii}^-$ for all i
  - Equal external competition rates: $a_O^- = a_{ij}^-$ for all i, j, $i \neq j$
  - Common migration rates: $a^+ = a_{ij}^+$ for i, j = 1, 2

- Set $\beta > \mu$ so system does not inevitably die out.

Results for N = 2 Boxes

- System may have up to 4 distinct non-negative singular points

All 4 are real and non-negative if

(*) $a_O^- > a_I^-$ and $\beta - \mu > 2a^+ \frac{a_O^- + a_I^-}{a_O^- - a_I^-}$

1) Trivial equilibrium at (0,0), unstable if $\beta > \mu$.
2) $\left(\frac{\beta-\mu}{a_I^- + a_O^-}, \frac{\beta-\mu}{a_I^- + a_O^-}\right)$, which always exists.

Results for N = 2 Boxes

3) $$\left(\frac{\beta-\mu-2a^+}{2a_I^-} + \frac{\sqrt{(\beta-\mu-2a^+)^2(a_O^- - a_I^-)^2 - 4a_I^- a^+(a_O^- - a_I^-)(\beta-\mu-2a^+)}}{2a_I^-(a_O^- - a_I^-)}, \quad \frac{\beta-\mu-2a^+}{2a_I^-} - \frac{\sqrt{(\beta-\mu-2a^+)^2(a_O^- - a_I^-)^2 - 4a_I^- a^+(a_O^- - a_I^-)(\beta-\mu-2a^+)}}{2a_I^-(a_O^- - a_I^-)} \right)$$

4) $$\left(\frac{\beta-\mu-2a^+}{2a_I^-} - \frac{\sqrt{(\beta-\mu-2a^+)^2(a_O^- - a_I^-)^2 - 4a_I^- a^+(a_O^- - a_I^-)(\beta-\mu-2a^+)}}{2a_I^-(a_O^- - a_I^-)}, \quad \frac{\beta-\mu-2a^+}{2a_I^-} + \frac{\sqrt{(\beta-\mu-2a^+)^2(a_O^- - a_I^-)^2 - 4a_I^- a^+(a_O^- - a_I^-)(\beta-\mu-2a^+)}}{2a_I^-(a_O^- - a_I^-)} \right)$$

Points 3 and 4 are stable equilibria.

Point 2 is a saddle point and not stable.

Results for N = 2 Boxes

- If condition (*) is not satisfied, point 2 is the only non-trivial equilibrium and it is stable.
- Point 2 is same equilibrium for each box as for 1-box model.
- Note existence of equilibria 3 and 4 depends on a^+ small enough, i.e., low migration rate.

  Contrary to what one might assume, that low migration would keep the 1-box equilibrium stable.

Results for N = 3 Boxes

- Results are similar to N= 2

- In particular, 2 equilibria always exist
  1) Trivial equilibrium at (0,0,0), unstable if $\beta > \mu$.
  2) $\left(\frac{\beta-\mu}{a_I^- + 2a_O^-}, \frac{\beta-\mu}{a_I^- + 2a_O^-}, \frac{\beta-\mu}{a_I^- + 2a_O^-} \right)$

- If $a_O^- = 0$, then point 2 is only non-trivial non-negative equilibrium

- Otherwise, under additional conditions, including sufficiently low migration between boxes, multiple non-negative equilibria can occur.

Ergodicity for *N* boxes

- Finally, we can establish geometric ergodicity
- We create $\{X_n\}_{n=0}^{\infty}$ on $(\mathbb{Z}_+)^N$, the embedded discrete time r.w. associated with our continuous r.w.

Set

$$c(\mathbf{x}) = \sum_{i=1}^{N} \left(\beta_i + \mu_i + \frac{a_{ii}^{-}}{L} x_i \right) x_i + \sum_{i,j=1, i \neq j}^{N} a_{ij}^{+} x_i$$

Ergodicity for *N* boxes

Transition probabilities:

for $\mathbf{x}, \mathbf{y} \in (\mathbb{Z}_+)^N$, $\mathbf{x} \neq \mathbf{0}$

$$P(\mathbf{x}, \mathbf{y}) = \frac{1}{c(\mathbf{x})} \cdot \begin{cases} \beta_i x_i & \text{if } \mathbf{y} = \mathbf{x} + e_i, i = 1, \ldots, N \\ \mu_i x_i + \frac{a_{ii}^-}{L} x_i^2 & \text{if } \mathbf{y} = \mathbf{x} - e_i, i = 1, \ldots, N \\ a_{ij}^+ x_i & \text{if } \mathbf{y} = \mathbf{x} - e_i + e_j, i \neq j \\ 0 & \text{otherwise} \end{cases}$$

for $\mathbf{x} = \mathbf{0}$

$$P(\mathbf{x}, \mathbf{y}) = \begin{cases} \frac{1}{N} & \text{if } \mathbf{y} = \mathbf{0} + e_i, i = 1, \ldots, N \\ 0 & \text{otherwise} \end{cases}$$

Ergodicity for *N* boxes

Theorem. A r.w. with the above transition probabilities is geometrically ergodic. That is, it is positively recurrent with exponential convergence to a stable distribution.

Ergodicity for *N* boxes

Method of proof:

Sufficient condition:

$$\sum_{\mathbf{y}\in(\mathbb{Z}_+)^N} P(\mathbf{x},\mathbf{y})V(\mathbf{y}) \leq \lambda V(\mathbf{x}) + b\mathbb{1}_B(\mathbf{x})$$

for Lyapunov function $V(x) \geq 1$

bounded set *B*, constants $b < \infty$, $\lambda < 1$

Lyapunov function: $V(\mathbf{x}) = \alpha^{||\mathbf{x}||_1}$ with $\alpha > 1$

We show α, λ, b, and B can be chosen such that the condition is met

Ergodicity for *N* boxes

Set external competition $a_{ij}^{-} = 0$ for $i \neq j$

Set symmetric conditions: for all *i*, $\beta_i \equiv \beta$,

$\mu_i \equiv \mu$, $a_{ij}^{+} \equiv a^{+}$, $a_{ii}^{-} \equiv a_I^{-}$, $\beta > \mu$

Then, drift vector for this r.w. $\overrightarrow{\Delta x} := P\vec{x} - \vec{x} = 0$

has at least 2 equilibria: 0 and $\vec{x}$ where

for all components *i*: $\frac{x_i}{L} = \frac{\beta - \mu}{a_I^{-}}$

Matches equilibria determined for *N* = 1, 2, 3.