Do Financial Returns Have Finite or Infinite Variance? A Paradox and an Explanation

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Financial Returns

Let P_t be the price for some financial asset at time t.

Assume that we observe the prices at evenly spaced times.

Log Returns:

$$R_{1,t} = \log\left(\frac{P_t}{P_{t-1}}\right) = \log(P_t) - \log(P_{t-1}).$$

Common Models For Returns

The most common models for financial returns are:

- Gaussian Distributions
- Infinite variance stable distributions
- Distributions with regularly varying tails

Stable Distributions

Definition

Let $X_1, X_2, \ldots$ be iid d-dimensional random vectors with distribution μ. μ is said to be ***stable*** *if for any n there are nonrandom constants $a_n > 0$ and $b_n \in \mathbb{R}^d$ such that*

$$a_n (X_1 + X_2 + \ldots + X_n - b_n) \stackrel{d}{=} X_1.$$

It turns out that $a_n = n^{-1/\alpha}$ for some $\alpha \in (0, 2]$, and we say that μ is α-stable.

A distribution is Gaussian if and only if it is 2-stable.

If we can take $b_n \equiv 0$ then we say that the distribution is **strictly stable**.

Generalized Central Limit Theorem

Let X be a d-dimensional random vector and let $X_1, X_2, \dots$ be iid copies of X. If there are constants $a_n > 0$ and $b_n \in \mathbb{R}^d$

$$a_n (X_1 + \cdots + X_n - b_n) \stackrel{d}{\to} Y,$$

then Y has a stable distribution.

In this case, we say that X belongs to the **domain of attraction** of Y.

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Aggregating Returns

If we add log returns of a certain frequency together, we get log returns of a lower frequency.

For example

$$\begin{aligned} R_{2,t} &= \log\left(\frac{P_t}{P_{t-2}}\right) = \log\left(\frac{P_t}{P_{t-1}}\right) + \log\left(\frac{P_{t-1}}{P_{t-2}}\right) \\ &= R_{1,t} + R_{1,t-1}. \end{aligned}$$

Log returns at a particular frequency are sums of log returns at higher frequencies.

Models of Returns

This suggests that at large aggregation levels, financial returns are well modeled by either Gaussian distributions or infinite variance stable distributions.

However, at small aggregation levels, we need to assume something about the underlying distribution.

It is most often assumed that at all aggregation levels, the distribution has regularly varying tails.

Regularly Varying Random Variables

Recall that a one-dimensional random variable X is said to have a **regularly varying right tail** with tail index $\alpha > 0$ if

$$P(X > x) = x^{-\alpha} L(x), \quad x > 0,$$

where L is a slowly varying at infinity function.

A similar definition applies to the regular variation of the left tail.

We will refer to the smaller of the two tail indices as the tail index.

Tail Index of Returns

What is the range of the tail index α in financial returns?

- If $0 < \alpha < 1$ then the absolute mean is infinite.
- If $1 < \alpha < 2$ then the mean is finite, but the variance is infinite.
- If $\alpha > 2$ then the variance is finite.

Is $\alpha > 2$?

Rachev and Mittnik (2000) find infinite variance and thus $\alpha < 2$.

Blattberg and Gonedes (1974) and Lau, Lau, and Wingender (1990) report finite variance, hence $\alpha > 2$.

Tail Paradox

Most confusing, and paradoxical:

The empirically estimated tail index appears to be lower for more frequent returns and higher for less frequent returns; see Gencay et al. (2001).

Thus one may find that daily (or more frequent) returns have an infinite variance, while weekly (or less frequent) returns have finite variance.

Tail Paradox

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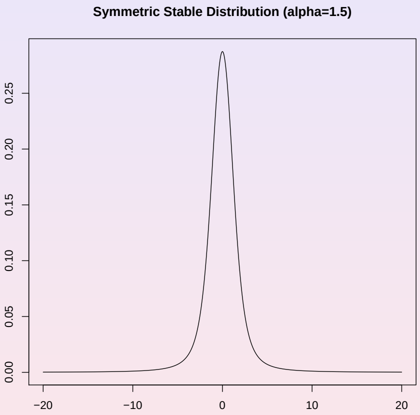
We resolve this paradox by proposing that the tails of returns are not actually regularly varying, but that they have what we term "tempered heavy tails".

Tempered Heavy Tailed Distributions

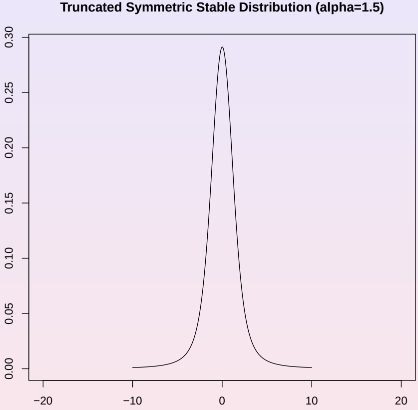
A probability distribution has **tempered heavy tails** if it can be obtained by modifying a distribution with regularly varying tails to make the tails lighter (that is to “temper” them).

- Tempering restricts the range where the regularly varying tails apply.
- Tempering can be done in a number of ways.

Example: Start With Stable Distribution



Then Truncate Its Tails



Normalizing constant: 1.0135.

Examples

Several tempered heavy tailed models have been successfully used in the finance, physics, and biostatistics.

- Truncated Lévy Flights, see Mantegna and Stanley (1994)
- Smoothly Truncated Lévy Flights, see Koponen (1995) and Tweedie (1984)
- Tempered Stable Distributions, see Rosiński (2007) and Grabchak (2016)
- Generalized Tempered Stable Distributions, see Rosiński and Sinclair (2010)

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Question: Why do stable-like distributions with light tails tend to come up in applications?

Regularly Varying Distribution: Symmetric Pareto

Consider the symmetric Pareto distribution with density

$$C|x|^{-1-\alpha}1_{b\leq|x|}dx$$

for $b, \alpha > 0$.

For $\alpha \in (0, 2)$, this is in the domain of attraction on an α-stable.

Truncated Symmetric Pareto

Now we truncate this distribution at some large T and the density becomes

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We can simulate this by using the accept-reject algorithm.

Simulate a Symmetric Pareto X

accept it if $|X| \leq T$

otherwise iterate

Truncated Symmetric Pareto

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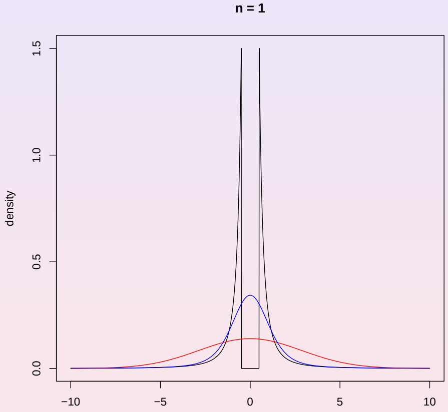
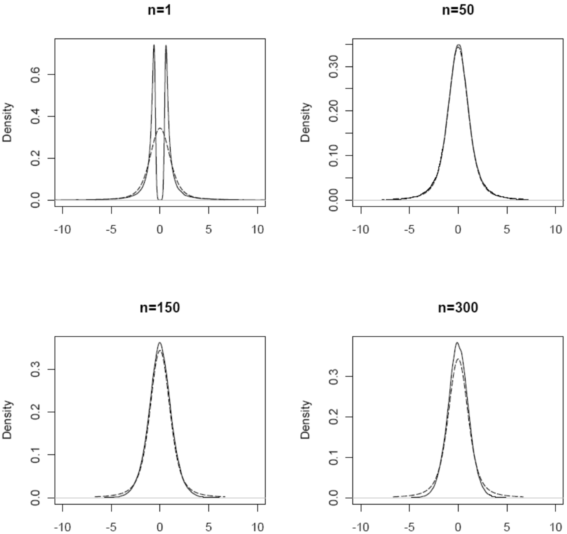
- For large enough n, the left side is approximately stable.
- However, the Y_i's are in the domain of attraction of the Gaussian.

Truncated Symmetric Pareto

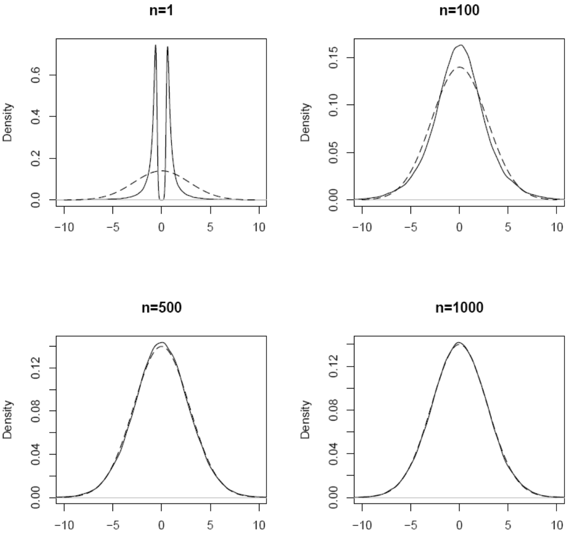
For concreteness take $\alpha = 1.5$, $b = .5$, and $T = 70$.



Sums of Truncated Symmetric Paretos and Approximating Stable



Sums of Truncated Symmetric Paretos and Approximating Gaussian



“Fake” Central Limit Behavior

We base a mathematical explanation of this “fake” CLT behavior on the Prelimit Theorems of Klebanov, Rachev, and Szekeley (1999).

The approach is:

- introduce a metric on the space of probability measures
- bound the distance between a stable distribution and a scaled sum of n iid random vectors

Probability Metric

Let h be a probability density on $\mathbb{R}^d$. For two d-dimensional random vectors X and Y define

$$K_h(X, Y) = \sup_{x \in \mathbb{R}^d} \left| F_X \star h(x) - F_Y \star h(x) \right| .$$

If h satisfies a Lipschitz condition and the corresponding characteristic function does not vanish, then K_h metrizes weak convergence on $\mathbb{R}^d$.

Distance Between Probability Distributions

Let X, Y be d-dimensional random vectors. For any $c, \gamma \geq 0$ define the following distance between the distributions of X and Y:

$$d_{c,\gamma}(X, Y) = \sup_{|z| \geq c} \frac{|\hat{\mu}_X(z) - \hat{\mu}_Y(z)|}{|z|^\gamma}.$$

If Y is a strictly α-stable random vector and X is a random vector such that

$$d_{0,\gamma}(X, Y) < \infty$$

for some $\gamma > \alpha$ then X is in the domain of attraction of Y.

Prelimit Theorem

Theorem

Let h be a probability density on $\mathbb{R}^d$ *such that* $h \in Lip_{M_h}$.

Let $X_1, X_2, \ldots$ *be iid d-dimensional random vectors, and let*

$$S_n = n^{-1/\alpha} \sum_{j=1}^{n} X_j.$$

If Y is a strictly α*-stable (*$\alpha \in (0,2]$*) random vector then for any* $\gamma > \alpha$

$$K_h(S_n, Y) \leq \inf_{a,\Delta>0} \left\{ \frac{d_{\Delta n^{-1/\alpha},\gamma}(X_1, Y)}{n^{\gamma/\alpha-1}} \frac{2^{\gamma+1}(a\sqrt{d})^{\gamma+d}}{\pi^{d/2}\Gamma(d/2)(\gamma+d)} + \frac{2}{\pi^d}[\Delta \wedge (2a)]^d + M_h \frac{12d}{\pi a} \right\}$$

Example: Truncated Symmetric Pareto

Let X be a truncated symmetric Pareto with parameters $\alpha \in (0, 2)$, $b = 1$, and $T > 0$. Let $X_1, X_2, \ldots$ be iid copies of X and let

$$S_n = n^{-1/\alpha} \left(X_1 + X_2 + \cdots + X_n \right).$$

Let Y be the stable distribution in whose domain of attraction the "untrunctated" symmetric Pareto would be in.

Example: Truncated Symmetric Pareto

For any $\gamma \in (\alpha, (2\alpha) \wedge 2)$ there is a constant $C = C(\gamma, \alpha)$ such that

$$\begin{aligned} K_h(S_n, Y) &\leq C \left(nT^{-\alpha}\right)^{1/(2(\gamma+1))} \\ &\quad + C \left(n^{-1} T^{\alpha}\right)^{1/2} n^{-(\gamma/\alpha - 1)} \end{aligned}$$

Back To Returns

- Financial returns are really sums
- If they have regularly varying tails, then the tail index should be the same at all aggregation levels.
- Since the tail index seems to increase, this suggests that the distributions are not regularly varying.

Back To Returns

- Instead we propose tempered heavy tailed distributions.
- Financial markets have built-in mechanisms designed to limit the fluctuation of prices.
- The prelimit theorem explains why sums of such distributions may appear to be stable at certain aggregation levels before ultimately converging to the Guassian.
- This suggests the need for “stable-like” distributions, but with lighter trails.

Bibliography

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What is the effect of the smoothing?