Stability of densities for perturbed Diffusions and Markov Chains.

V. Konakov[1], A. Kozhina[1] and S. Menozzi[2,1]

[1]Higher School of Economics, Moscow, Russia
[2]Higher School of Economics, Moscow, Russia and Université d'Evry Val d'Essonne, Evry, France.

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Contents

For a fixed given deterministic final horizont $T > 0$ let us consider multidimensional SDE and its perturbed version:

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dW_t, \ t \in [0, T], \tag{1.1}$$

$$dX_t^{(\varepsilon)} = b_\varepsilon(t, X_t^{(\varepsilon)})dt + \sigma_\varepsilon(t, X_t^{(\varepsilon)})dW_t, t \in [0, T], \tag{1.2}$$

where $b, b_\varepsilon : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d, \ \sigma, \sigma_\varepsilon : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d \otimes \mathbb{R}^d$ are bounded coefficients that are measurable in time and Hölder continuous in space.

Also, $a(t, x) := \sigma\sigma^*(t, x), a_\varepsilon(t, x) := \sigma_\varepsilon\sigma_\varepsilon^*(t, x)$ are assumed to be uniformly elliptic.

Assumptions

$\varepsilon > 0$ is fixed and the constants appearing in the assumptions do not depend on ε.

(A1) **(Boundedness)**. $\exists K_1, K_2 > 0$ s.t.

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} |b(t,x)| + \sup_{(t,x)\in[0,T]\times\mathbb{R}^d} |b_\varepsilon(t,x)| \le K_1,$$
$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} |\sigma(t,x)| + \sup_{(t,x)\in[0,T]\times\mathbb{R}^d} |\sigma_\varepsilon(t,x)| \le K_2.$$

(A2) **(UE)**. a, a_ε are uniformly elliptic, i.e. there exists $\Lambda \ge 1,\ \forall (t,x,\xi) \in [0,T] \times (\mathbb{R}^d)^2$,

$$\begin{aligned} \Lambda^{-1}|\xi|^2 &\le \langle a(t,x)\xi,\xi\rangle &\le \Lambda|\xi|^2 \\ \Lambda^{-1}|\xi|^2 &\le \langle a_\varepsilon(t,x)\xi,\xi\rangle &\le \Lambda|\xi|^2. \end{aligned}$$

(A3) **(Hölder continuity in space)**. For some $\gamma \in (0,1]$, $\kappa < \infty$, for all $t \in \mathbb{R}_+$,

$$|\sigma(t,x) - \sigma(t,y)| + |\sigma_\varepsilon(t,x) - \sigma_\varepsilon(t,y)| \quad \le \quad \kappa\,|x-y|^\gamma .$$

Assumptions

Set for $\varepsilon > 0$:

$$\Delta_{\varepsilon,b,\infty} := \sup_{(t,x)\in[0,T]\times\mathbb{R}^d} \{|b(t,x) - b_\varepsilon(t,x)|\},$$

$$\Delta_{\varepsilon,b,q} := \sup_{t\in[0,T]} \|b(t,.) - b_\varepsilon(t,.)\|_{L^q(\mathbb{R}^d)},\ \forall q \in (1,+\infty).$$

Since $\sigma, \sigma_\varepsilon$ are both γ-Hölder continuous, see (**A3**) we also define

$$\Delta_{\varepsilon,\sigma,\gamma} := \sup_{u\in[0,T]} |\sigma(u,.) - \sigma_\varepsilon(u,.)|_\gamma,$$

where for $\gamma \in (0,1]$, $|.|_\gamma$ stands for the usual Hölder norm in space on $C_b^\gamma(\mathbb{R}^d, \mathbb{R}^d \otimes \mathbb{R}^d)$:

$$|f|_\gamma := \sup_{x\in\mathbb{R}^d} |f(x)| + [f]_\gamma,\ [f]_\gamma := \sup_{x\neq y,(x,y)\in(\mathbb{R}^d)^2} \frac{|f(x) - f(y)|}{|x-y|^\gamma}.$$

The previous control in particular implies $\forall (u,x,y) \in [0,T] \times (\mathbb{R}^d)^2$:

$$|a(u,x) - a(u,y) - a_\varepsilon(u,x) + a_\varepsilon(u,y)| \leq 2(K_2 + \kappa)\Delta_{\varepsilon,\sigma,\gamma}|x-y|^\gamma.$$

We eventually set for $q \in (1,+\infty]$,

$$\Delta_{\varepsilon,\gamma,q} := \Delta_{\varepsilon,\sigma,\gamma} + \Delta_{\varepsilon,b,q}. \tag{1.3}$$

Stability for diffusions

For a given $c > 0$ and for all $(u,z) \in \mathbb{R}^{+*} \times \mathbb{R}^d$ we denote:
$p_c(u,z) := \frac{c^{d/2}}{(2\pi u)^{d/2}} \exp(-c\frac{|z|^2}{2u})$.

Theorem 1 (Stability Control for diffusions)

*Fix $\varepsilon > 0$ and a final deterministic time horizon $T > 0$. Under (**A**) and for $q > d$, there exist $C := C(q) \geq 1$, $c := c(q) \in (0,1]$ s.t. for all $0 \leq s < t \leq T, (x,y) \in (\mathbb{R}^d)^2$:*

$$p_c(t-s, y-x)^{-1}|(p - p_\varepsilon)(s,t,x,y)| \leq C\Delta_{\varepsilon,\gamma,q}, \tag{1.4}$$

where $p(s,t,x,.), p_\varepsilon(s,t,x,.)$ respectively stand for the transition densities at time t of equations (1.1), (1.2) starting from x at time s.

Model Sensitivity for Option Prices.

Assume that the (log)-price of an asset is given by the dynamics in (1.1).

Price of an option: $\mathbb{E}[f(\exp(X_T^{t,x}))]$ up to an additional discounting factor. f is the pay-off function.

Theorem 1 allows to specifically quantify how a perturbation of the coefficients impacts the option prices.

For a given $\varepsilon > 0$

$$|\mathbb{E}_\varepsilon(t, T, x, f)| := |\mathbb{E}[f(\exp(X_T^{t,x}))] - \mathbb{E}[f(\exp(X_T^{t,x,(\varepsilon)}))]| \\ \leq C\Delta_{\varepsilon,\gamma,q} \int_{\mathbb{R}^d} f(\exp(y))p_c(t, T, x, y)dy.$$

Parametrix

Assume that the solution $(X_t^{s,x})_{t\geq s}$ of (1.1) starting from x at time s has for all $t > s$ a smooth density $p(s,t,x,.)$.

For all $(s,x) \in [0,T] \times \mathbb{R}^d,\ t \geq s$ we introduce the following Gaussian inhomogeneous process with spatial variable frozen at given point $y \in \mathbb{R}^d$:

$$\tilde{X}_t^y = x + \int_s^t \sigma(u,y) dW_u.$$

Its density $\tilde{p}^y$ readily satisfies the Kolmogorov Backward equation:

$$\begin{cases} \partial_u \tilde{p}^y(u,t,z,y) + \tilde{L}_u^y \tilde{p}^y(u,t,z,y) = 0,\ s \leq u < t, z \in \mathbb{R}^d, \\ \tilde{p}^y(u,t,.,y) \underset{u\uparrow t}{\to} \delta_y(.), \end{cases} \tag{1.5}$$

where for all $\varphi \in C_0^2(\mathbb{R}^d, \mathbb{R}), z \in \mathbb{R}^d$:

$$\tilde{L}_u^y \varphi(z) = \frac{1}{2} \mathrm{Tr}\left(\sigma\sigma^*(u,y) D_z^2 \varphi(z)\right),$$

stands for the generator of $\tilde{X}^y$ at time u.

Since we have assumed the density of X to be smooth, it must satisfy the Kolmogorov forward equation (see [Dyn65]) for a given starting point $x \in \mathbb{R}^d$

$$\begin{cases} \partial_u p(s,u,x,z) = L_u^* p(s,u,x,z) = 0, \ s < u \le t, z \in \mathbb{R}^d, \\ p(s,u,x,.) \underset{u \downarrow s}{\to} \delta_x(.), \end{cases} \tag{1.6}$$

where L_u^* stands for the *formal* adjoint (which is well defined because the coefficients in (1.1) are smooth) of the generator of (1.1) which for all $\varphi \in C_0^2(\mathbb{R}^d, \mathbb{R}), z \in \mathbb{R}^d$ writes:

$$L_u \varphi(z) = \frac{1}{2} \mathrm{Tr}\left(\sigma\sigma^*(u,z) D_z^2 \varphi(z)\right) + \langle b(u,z), D_z \varphi(z)\rangle.$$

Parametrix

Equations (1.5), (1.6) yield the formal expansion below which is initially due to [MS67]

$$\begin{aligned}(p-\tilde p^y)(s,t,x,y) &= \int_s^t du\partial_u \int_{\mathbb{R}^d} dzp(s,u,x,z)\tilde p^y(u,t,z,y)\\ &= \int_s^t du \int_{\mathbb{R}^d} dzp(s,u,x,z)(L_u-\tilde L_u^y)\tilde p^y(u,t,z,y). \qquad (1.7)\end{aligned}$$

Let us now introduce the notation

$$f \otimes g(s,t,x,y) = \int_s^t du \int_{\mathbb{R}^d} dzf(s,u,x,z)g(u,t,z,y)$$

for the time-space convolution.

Let us define $\tilde p(s,t,x,y) := \tilde p^y(s,t,x,y)$ - the density of the frozen process at the final point and observe it at *that specific* point.

Parametrix

We now introduce the parametrix kernel:

$$H(s,t,x,y) := (L_s - \tilde{L}_s)\tilde{p}(s,t,x,y) := (L_s - \tilde{L}_s^y)\tilde{p}^y(s,t,x,y). \quad (1.8)$$

With those notations equation (1.7) rewrites:

$$(p - \tilde{p})(s,t,x,y) = p \otimes H(s,t,x,y).$$

This yields to iterated convolutions of the kernel and leads to the formal expansion:

$$p(s,t,x,y) = \sum_{r=0}^{\infty} \tilde{p} \otimes H^{(r)}(s,t,x,y), \quad (1.9)$$

where $\tilde{p} \otimes H^{(0)} = \tilde{p}, H^{(r)} = H \otimes H^{(r-1)},\ r \geq 1.$

Proposition 1

Under the sole assumption (A), for $t > s$, the density of $X_t^{x,s}$ solving (1.1) *exists and can be written as in* (1.9).

Parametrix

We consider the parametrix series:

$$p(s,t,x,y) =$$
$$\widetilde{p}(s,t,x,y) + \sum_{r=1}^{\infty} \int_s^t du \int_{\mathbb{R}^d} \widetilde{p}(s,u,x,z) H^{(r)}(u,t,z,y) dz,$$
$$p_\varepsilon(s,t,x,y) =$$
$$\widetilde{p}_\varepsilon(s,t,x,y) + \sum_{r=1}^{\infty} \int_s^t du \int_{\mathbb{R}^d} \widetilde{p}_\varepsilon(s,u,x,z) H_\varepsilon^{(r)}(u,t,z,y) dz,$$

Estimations

Following the method considered in [KM00] one can prove the estimates below.

$$\forall 0 \le u < t \le T,\ (z,y) \in (\mathbb{R}^d)^2,$$
$$\left|D_z^\alpha \widetilde{p}(u,t,z,y)\right| \le \frac{c_1}{(t-u)^{|\alpha|/2}} p_c(t-u, y-z), \qquad (1.10)$$

From (1.10), the boundedness of the drift and the Hölder continuity in space of the diffusion matrix: $\exists c_1 \ge 1,\ c \in (0,1]$,

$$|H(u,t,z,y)| \le \frac{c_1(1 \vee T^{(1-\gamma)/2})}{(t-u)^{1-\gamma/2}} p_c(t-u, z-y). \qquad (1.11)$$

It follows by induction that:

$$|\widetilde{p} \otimes H^{(r)}(s,t,x,y)| \le \frac{((1 \vee T^{(1-\gamma)/2})c_1)^{r+1} \left[\Gamma(\frac{\gamma}{2})\right]^r}{\Gamma(1 + r\frac{\gamma}{2})} p_c(t-s, y-x)(t-s)^{\frac{r\gamma}{2}}.$$

$$p(s,t,x,y) \le c_1 \exp((1 \vee T^{(1-\gamma)/2})c_1[(t-s)^{\gamma/2}])p_c(t-s, y-x).$$

Step 1.

Let us consider the difference between two parametrix expansions:

$$|p(s,t,x,y) - p_\varepsilon(s,t,x,y)| =$$

$$\leq |(\tilde{p} - \tilde{p}_\varepsilon)(s,t,x,y)| + |\sum_{r=1}^{\infty} \tilde{p} \otimes H^{(r)}(s,t,x,y) - \sum_{r=1}^{\infty} \tilde{p}_\varepsilon \otimes H_\varepsilon^{(r)}(s,t,x,y)|.$$

Lemma 1 (Difference of the first terms and their derivatives)

*Under **(A)**, there exist* $c_1 \geq 1,\ c \in (0,1]$ *s.t. for all* $0 \leq s < t,\ (x,y) \in (\mathbb{R}^d)^2$ *and all multi-index* $\alpha,\ |\alpha| \leq 4$,

$$|D_x^\alpha \tilde{p}(s,t,x,y) - D_x^\alpha \tilde{p}_\varepsilon(s,t,x,y)| \leq \frac{c_1}{(t-s)^{|\alpha|/2}} \Delta_{\varepsilon,\sigma,\gamma} p_c(t-s, y-x).$$

The key point of the proof consists in using multidimentional Taylor expansion with respect to covariance and mean functions.

End of the proof

We consider the difference between "sums". Let us estimate:

$$|\sum_{r=1}^{\infty}\widetilde{p}\otimes H^{(r)}(s,t,x,y)-\sum_{r=1}^{\infty}\widetilde{p}_{\varepsilon}\otimes H_{\varepsilon}^{(r)}(s,t,x,y)|;$$

Lemma 2 (Difference of the iterated kernels)

For all $0\le s<t\le T,\ (x,y)\in(\mathbb{R}^d)^2$ *and for all* $q\in(d,+\infty]$, $r\in\mathbb{N}$*:*

$$|(\tilde{p}\otimes H^{(r)}-\tilde{p}_{\varepsilon}\otimes H_{\varepsilon}^{(r)})(s,t,x,y)|$$
$$\le(r+1)\Delta_{\varepsilon,\gamma,q}\frac{\bar{C}^{r+1}\left[\Gamma(\frac{\gamma}{2}\wedge\alpha(q))\right]^r}{\Gamma(1+r(\frac{\gamma}{2}\wedge\alpha(q)))}p_c(t-s,y-x)(t-s)^{r(\frac{\gamma}{2}\wedge\alpha(q))},$$

where $\alpha(q):=\frac{1}{2}\left(1-\frac{d}{q}\right)$.

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Stability of transition densities for MC

Our stability results will also apply to two Markov Chains with respective dynamics:

$$\begin{aligned} Y_{t_{k+1}} &= Y_{t_k} + b(t_k, Y_{t_k})h + \sigma(t_k, Y_{t_k})\sqrt{h}\xi_{k+1},\ Y_0 = x, \\ Y^{(\varepsilon)}_{t_{k+1}} &= Y^{(\varepsilon)}_{t_k} + b_\varepsilon(t_k, Y^{(\varepsilon)}_{t_k})h + \sigma_\varepsilon(t_k, Y^{(\varepsilon)}_{t_k})\sqrt{h}\xi_{k+1},\ Y^{(\varepsilon)}_0 = x, \end{aligned} \quad (2.1)$$

where $h > 0$ is a given time step, for which we denote for all $k \geq 0,\ t_k := kh$ and the $(\xi_k)_{k\geq 1}$ are i.i.d. centered random variables.

We introduce two kinds of innovations in (2.1):

(I_G) The i.i.d. random variables $(\xi_k)_{k\geq 1}$ are Gaussian, with law $\mathcal{N}(0, I_d)$. In that case the dynamics in (2.1) correspond to the Euler discretization of equations (1.1) and (1.2).

($I_{P,M}$) For a given integer $M > 2d + 5 + \gamma$, the innovations $(\xi_k)_{k\geq 1}$ are centered and have C^5 density f_ξ which has, together with its derivatives up to order 5, at most polynomial decay of order M. Namely, for all $z \in \mathbb{R}^d$ and multi-index $\nu, |\nu| \leq 5$:

$$|D^\nu f_\xi(z)| \leq CQ_M(z),$$

where we denote for all
$r > d,\ z \in \mathbb{R}^d,\ \ Q_r(z) := c_r \frac{1}{(1+|z|)^r},\ \int_{\mathbb{R}^d} dz Q_r(z) = 1.$

Stability for MC

Fix $\varepsilon > 0$ and a final deterministic time horizon $T > 0$. For $h = T/N,\ N \in \mathbb{N}^*$, we set for $i \in \mathbb{N},\ t_i := ih$. Under (**A**), assuming that either I_G or $I_{P,M}$ holds, and for $q > d$:

Theorem 2 (Stability Control for Markov Chains)

$\exists C := C(q) \geq 1, c := c(q) \in (0,1]$ *s.t. for all* $0 \leq t_i < t_j \leq T, (x,y) \in (\mathbb{R}^d)^2$*:*

$$\chi_c(t_j - t_i, y - x)^{-1} |(p^h - p^h_\varepsilon)(t_i, t_j, x, y)| \leq C\Delta_{\varepsilon,\gamma,q}, \qquad (2.2)$$

where $p^h(t_i, t_j, x, .), p^h_\varepsilon(t_i, t_j, x, .)$ *respectively stand for the transition densities at time* t_j *of the Markov Chains* Y *and* $Y^{(\varepsilon)}$ *in* (2.1) *starting from* x *at time* t_i*.*

- *If* I_G *holds:* $\chi_c(t_j - t_i, y - x) := p_c(t_j - t_i, y - x)$, *with* p_c *as in Theorem 1.*
- *If* $I_{P,M}$ *holds:* $\chi_c(t_j - t_i, y - x) := \frac{c^{d/2}}{(t_j - t_i)^{d/2}} Q_{M-(d+5+\gamma)}\left(\frac{|y-x|}{\{(t_j-t_i)\}^{1/2}/c}\right)$.

Denote by $(Y_{t_j}^{t_i,x})_{j\ge i}$ the Markov chain with dynamics (2.1) starting from x at time t_i. Observe first that if the innovations $(\xi_k)_{k\ge 1}$ have a density then so does the chain at time t_k. Let us now introduce its

generator at time t_i, i.e. for all $\varphi \in C_0^2(\mathbb{R}^d,\mathbb{R}),\ x\in\mathbb{R}^d$:

$$L_{t_i}^h\varphi(x) := h^{-1}\mathbb{E}[\varphi(Y_{t_{i+1}}^{t_i,x}) - \varphi(x)].$$

In order to give a representation of the density of $p^h(t_i,t_j,x,y)$ of $Y_{t_j}^{t_i,x}$ at point y for $j>i$, we introduce similarly to the continuous case, the Markov chain (or inhomogeneous random walk) with coefficients frozen in space at y. For given $(t_i,x)\in[0,T]\times\mathbb{R}^d,\ t_j\ge t_i$ we set:

$$\tilde{Y}_{t_j}^{t_i,x,y} := x + h^{1/2}\sum_{k=i}^{j-1}\sigma(t_k,y)\xi_{k+1},$$

and denote its density $\tilde{p}^{h,y}(t_i,t_j,x,.)$. Its generator at time t_i writes for all $\varphi\in C_0^2(\mathbb{R}^d,\mathbb{R}), x\in\mathbb{R}^d$:

$$\tilde{L}_{t_i}^{h,y}\varphi(x) = h^{-1}\mathbb{E}[\varphi(\tilde{Y}_{t_{i+1}}^{t_i,x,y}) - \varphi(x)].$$

Parametrix for MC

Using the notation $\tilde{p}^h(t_i, t_j, x, y) := \tilde{p}^{h,y}(t_i, t_j, x, y)$, we introduce now for $0 \le i < j \le N$ the *parametrix* kernel:

$$H^h(t_i, t_j, x, y) := (L^h_{t_i} - \tilde{L}^{h,y}_{t_i})\tilde{p}^h(t_i + h, t_j, x, y).$$

The discrete time convolution type operator $\otimes_h$ is defined by

$$f \otimes_h g(t_i, t_j, x, y) = \sum_{k=0}^{j-i-1} h \int_{\mathbb{R}^d} f(t_i, t_{i+k}, x, z) g(t_{i+k}, t_j, z, y) dz.$$

Also $g \otimes H^{h,(0)} = g$ and for all $r \ge 1$, $H^{h,(r)} := H^h \otimes_h H^{h,(r-1)}$ denotes the r-fold discrete convolution of the kernel H^h.

Proposition 2 (Parametrix Expansion for the Markov Chain)

Assume (A) *is in force. Then, for* $0 \le t_i < t_j \le T$,

$$p^h(t_i, t_j, x, y) = \sum_{r=0}^{j-i} \tilde{p}^h \otimes_h H^{h,(r)}(t_i, t_j, x, y).$$

Stability for MC. Main parts

Lemma 3 (Controls and Comparison of the densities and their derivatives)

There exist c, c_1 s.t. for all $0 \le t_i < t_j \le T,\ (x,y) \in (\mathbb{R}^d)^2$ and for all multi-index $\alpha,\ |\alpha| \le 4$:

$$|D_x^\alpha \tilde{p}^h(t_i,t_j,x,y)| + |D_x^\alpha \tilde{p}_\varepsilon^h(t_i,t_j,x,y)| \le \frac{1}{(t_j-t_i)^{|\alpha|/2}} \psi_{c,c_1}(t_j-t_i, y-x),$$

$$|D_x^\alpha \tilde{p}^h(t_i,t_j,x,y) - D_x^\alpha \tilde{p}_\varepsilon^h(t_i,t_j,x,y)| \le \frac{\Delta_{\varepsilon,\sigma,\gamma}}{(t_j-t_i)^{|\alpha|/2}} \psi_{c,c_1}(t_j-t_i, y-x),$$

where

- *Under I_G:* $\psi_{c,c_1}(t_j-t_i, y-x) := c_1 p_c(t_j-t_i, y-x)$,
- *Under $I_{P,M}$:* $\psi_{c,c_1}(t_j-t_i, y-x) := \frac{c_1}{(t_j-t_i)^{d/2}} Q_{M-d-5}\left(\frac{|y-x|}{(t_j-t_i)^{1/2}}\right)$.

Stability for MC. Kernels

Lemma 4 (Control of the One-Step Convolution for the Chain.)

There exists c_1, c s.t. for all $q = +\infty$ and for $0 \le t_k < t_j \le T, (z, y) \in (\mathbb{R}^d)^2$:

$$|(H^h - H^h_\varepsilon)(t_k, t_j, z, y)| \le \frac{\Delta_{\varepsilon,\gamma,\infty}}{(t_j - t_k)^{1-\gamma/2}} \Phi_{c,c_1}(t_j - t_k, z - y),$$

with

- $\Phi_{c,c_1}(t_j - t_k, z - y) = \psi_{c,c_1}(t_j - t_k, z - y)$ *under* I_G.
- $\Phi_{c,c_1}(t_j - t_k, z - y) = \psi_{c,c_1}(t_j - t_k, z - y) \left(1 + \frac{|z-y|}{(t_j - t_k)^{1/2}}\right)^\gamma$*, under* $I_{P,M}$*,*

where ψ_{c,c_1} is defined according to the assumptions on the innovations in Lemma 3.

Stability for MC. Summary

From the controls of Lemma 4 and following the strategy of Lemma 2, we will be led to consider convolutions of the previous type involving Γ functions. The above strategy thus yields the main result by induction.

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Let us consider the $\mathbb{R}^d \times \mathbb{R}^d-$ process:

$$\begin{cases} dX_t = b(X_t, Y_t)dt + \sigma(X_t, Y_t)dW_t, \\ dY_t = X_t dt, t \in [0, T], \end{cases} \tag{3.1}$$

and its *perturbed* version:

$$\begin{cases} dX_t^{(\varepsilon)} = b_\varepsilon(X_t^{(\varepsilon)}, Y_t^{(\varepsilon)})dt + \sigma(X_t^{(\varepsilon)}, Y_t^{(\varepsilon)})dW_t, \\ dY_t^{(\varepsilon)} = X_t^{(\varepsilon)} dt, t \in [0, T], \end{cases} \tag{3.2}$$

where $b, b_\varepsilon : \mathbb{R}^{2d} \to \mathbb{R}^d, \ \sigma, \sigma_\varepsilon : \mathbb{R}^{2d} \to \mathbb{R}^d \otimes \mathbb{R}^d$ are bounded coefficients that are measurable in time and Hölder continuous in space.. $a(x, y) := \sigma\sigma^*(x, y), a_\varepsilon(x, y) := \sigma_\varepsilon\sigma_\varepsilon^*(x, y)$ uniformly elliptic. Under these conditions there are exist weak solutions for (3.1) and (3.2).

Motivation

Such kind of processes appear in various applicative fields.

- **Finance**
  Asian option. X - the dynamics of the underlying asset , Y is involved in the option Payoff. Option's price - $\mathbb{E}_x[\psi(X_T, T^{-1}Y_T)]$, where $\psi(x, y) = (x - y)^+$ (resp.$(y - x)^+$).
- **Hamiltonian systems**
  For a given Hamilton function of the form $H(x, y) = V(y) + \frac{|x|^2}{2}$, where V is a potential and $\frac{|x|^2}{2}$ - the kinetic energy of a particle with unit mass, the associated stochastic Hamiltonian system would correspond to $b(X_s, Y_s) = -(\partial y V(Y_s) + F(X_s, Y_s)X_s)$ in (3.1), where F is a friction term.

Stability. Degenerate case

Theorem 3 (Stability Control for Degenerate Diffusions)

*Fix $T > 0$. Under (**A**), for $q \in (4d, +\infty]$, there exist $C := C(q) \geq 1, c \in (0, 1]$ such that for all $0 < t \leq T, ((x, y), (x', y')) \in (\mathbb{R}^{2d})^2$:*

$$|(p - p_\varepsilon)(t, (x, y), (x', y'))| \leq C\Delta_{\varepsilon,\gamma,q}\hat{p}_{c,K}(t, (x, y), (x', y')).$$

where $p(t, (x, y), (., .)), p_\varepsilon(t, (x, y), (., .))$ respectively stand for the transition at time t of equations (3.1), (3.2), starting from x, y at time 0.

$$\hat{p}_{c,K}(t, (x, y), (x', y'))$$
$$:= \frac{c^d 3^{d/2}}{(2\pi t^2)^d} \exp\left(-c\left[\frac{|x' - x|^2}{4t} + 3\frac{|y' - y - (x + x')t/2|^2}{t^3}\right]\right),$$

Bibliography

- E. B Dynkin.
  Markov Processes.
  Springer Verlag, 1965.
- V. Konakov and E. Mammen.
  Local limit theorems for transition densities of Markov chains converging to diffusions.
  Prob. Th. Rel. Fields, 117:551–587, 2000.
- H. P. McKean and I. M. Singer.
  Curvature and the eigenvalues of the Laplacian.
  J. Differential Geometry, 1:43–69, 1967.

Thank you for your attention!