Three Upsilon Transforms Related to Tempered Stable Distributions

Michael Grabchak

University of North Carolina Charlotte
Department of Mathematics and Statistics

XVII April International Academic Conference on Economic and Social Development

April 19, 2016

Classes of Measures

- Let $\mathfrak{M}$ be the collection of Borel measures on $\mathbb{R}^d$

Classes of Measures

- Let $\mathfrak{M}$ be the collection of Borel measures on $\mathbb{R}^d$
- Let $\mathfrak{M}_{sf0}$ be the collection of $M \in \mathfrak{M}$ such that M is σ-finite and
$$M(\{0\}) = 0.$$

Classes of Measures

- Let $\mathfrak{M}$ be the collection of Borel measures on $\mathbb{R}^d$
- Let $\mathfrak{M}_{sf0}$ be the collection of $M \in \mathfrak{M}$ such that M is σ-finite and
$$M(\{0\}) = 0.$$
- Let $\mathfrak{M}^0$ be all Lévy measures on $\mathbb{R}^d$, i.e. the collection of all $M \in \mathfrak{M}_{sf0}$ such that
$$\int_{|x|\le 1} |x|^2 M(\mathrm{d}x) + \int_{|x|>1} M(\mathrm{d}x) < \infty.$$

Classes of Lévy Measures

For $\alpha \in [0,2]$ let $\mathfrak{M}^\alpha \subset \mathfrak{M}^0$ and $M \in \mathfrak{M}^\alpha$ if

$$\int_{|x|\le 1} |x|^2 M(\mathrm{d}x) + \int_{|x|>1} |x|^\alpha M(\mathrm{d}x) < \infty.$$

Classes of Lévy Measures

For $\alpha \in [0,2]$ let $\mathfrak{M}^\alpha \subset \mathfrak{M}^0$ and $M \in \mathfrak{M}^\alpha$ if

$$\int_{|x|\le 1} |x|^2 M(\mathrm{d}x) + \int_{|x|>1} |x|^\alpha M(\mathrm{d}x) < \infty.$$

Let $\mathfrak{M}^{\log} \subset \mathfrak{M}^0$ where $M \in \mathfrak{M}^{\log}$ satisfies

$$\int_{|x|\le 1} |x|^2 M(\mathrm{d}x) + \int_{|x|>1} \log|x| M(\mathrm{d}x) < \infty.$$

Upsilon Transforms

Parameter: Let ρ be a nonzero σ-finite Borel measure on $(0, \infty)$.

A mapping $\Upsilon_\rho : \mathfrak{M}_{sf0} \mapsto \mathfrak{M}$ of the form

$$[\Upsilon_\rho M](B) = \int_0^\infty M(s^{-1}B)\rho(\mathrm{d}s), \quad M \in \mathfrak{M}_{sf0}, \ \ B \in \mathfrak{B}(\mathbb{R}^d)$$

is called an **upsilon transform with dilation measure** ρ.

Upsilon Transforms

Parameter: Let ρ be a nonzero σ-finite Borel measure on $(0, \infty)$.

A mapping $\Upsilon_\rho : \mathfrak{M}_{sf0} \mapsto \mathfrak{M}$ of the form

$$[\Upsilon_\rho M](B) = \int_0^\infty M(s^{-1}B)\rho(\mathrm{d}s), \quad M \in \mathfrak{M}_{sf0}, \ B \in \mathfrak{B}(\mathbb{R}^d)$$

is called an **upsilon transform with dilation measure** ρ.

Defined in Barndorff-Nielsen, Rosiński, Thorbjørnsen (2008)

Idea goes back, at least, to Jurek (1990).

Interpretation 1: Product Convolutions

Let ρ be a probability measure on $(0, \infty)$

Let M be a probability measure on $\mathbb{R}^d$ with $M(\{0\}) = 0$.

Interpretation 1: Product Convolutions

Let ρ be a probability measure on $(0, \infty)$

Let M be a probability measure on $\mathbb{R}^d$ with $M(\{0\}) = 0$.

If $X \sim \rho$ and $\vec{Y} \sim M$ are independent RVs then

Interpretation 1: Product Convolutions

Let ρ be a probability measure on $(0, \infty)$

Let M be a probability measure on $\mathbb{R}^d$ with $M(\{0\}) = 0$.

If $X \sim \rho$ and $\vec{Y} \sim M$ are independent RVs then

$$X\vec{Y} \sim \Upsilon_\rho(M).$$

Interpretation 2: Stochastic Integration

Let $\{X_t : t \geq 0\}$ be a Lévy process on $\mathbb{R}^d$ with Lévy measure M.

Interpretation 2: Stochastic Integration

Let $\{X_t : t \geq 0\}$ be a Lévy process on $\mathbb{R}^d$ with Lévy measure M.

This means that the characteristic function of X_t is of the form

$$e^{tC(z)}, \quad z \in \mathbb{R}^d,$$

where

$$\begin{aligned} C(z) &= -\frac{1}{2}\langle z, Az\rangle + i\langle b, z\rangle \\ &\quad + \int_{\mathbb{R}^d} \left(e^{i\langle x,z\rangle} - 1 - i\langle x, z\rangle 1_{|x|\leq 1} \right) M(\mathrm{d}x). \end{aligned}$$

Here A is a $d \times d$ covariance matrix, $b \in \mathbb{R}^d$, and $M \in \mathfrak{M}^0$.

Interpretation 2: Stochastic Integration

Let $\{X_t : t \geq 0\}$ be a Lévy process on $\mathbb{R}^d$ with Lévy measure M.

Interpretation 2: Stochastic Integration

Let $\{X_t : t \geq 0\}$ be a Lévy process on $\mathbb{R}^d$ with Lévy measure M.

Let ρ be a nonzero σ-finite Borel measure on $(0, \infty)$ and define

$$\eta_\rho(t) = \rho([t, \infty)), \quad t > 0$$

Assume that $\eta_\rho(t) < \infty$ for each $t > 0$ and define

$$\eta_\rho^*(t) = \inf\{s > 0 : \eta_\rho(s) \leq t\}, \quad t > 0.$$

Interpretation 2: Stochastic Integration

Let $\{X_t : t \geq 0\}$ be a Lévy process on $\mathbb{R}^d$ with Lévy measure M.

Let ρ be a nonzero σ-finite Borel measure on $(0, \infty)$ and define

$$\eta_\rho(t) = \rho([t, \infty)), \quad t > 0$$

Assume that $\eta_\rho(t) < \infty$ for each $t > 0$ and define

$$\eta_\rho^*(t) = \inf\{s > 0 : \eta_\rho(s) \leq t\}, \quad t > 0.$$

Let

$$Y = \int_0^{\eta_\rho(0)} \eta_\rho^*(t) \mathrm{d}X_t.$$

Interpretation 2: Stochastic Integration

Let $\{X_t : t \geq 0\}$ be a Lévy process on $\mathbb{R}^d$ with Lévy measure M.

If the integral

$$Y = \int_0^{\eta_\rho(0)} \eta_\rho^*(t)\mathrm{d}X_t.$$

exists then Y is infinitely divisible with Lévy measure $\Upsilon_\rho(M)$.

Interpretation 2: Stochastic Integration

Let $\{X_t : t \geq 0\}$ be a Lévy process on $\mathbb{R}^d$ with Lévy measure M.

If the integral

$$Y = \int_0^{\eta_\rho(0)} \eta_\rho^*(t) \mathrm{d}X_t.$$

exists then Y is infinitely divisible with Lévy measure $\Upsilon_\rho(M)$.

For the integral to exist, it is necessary that $\Upsilon_\rho(M) \in \mathfrak{M}^0$.

Interpretation 2: Stochastic Integration

Let $\{X_t : t \geq 0\}$ be a Lévy process on $\mathbb{R}^d$ with Lévy measure M.

If the integral

$$Y = \int_0^{\eta_\rho(0)} \eta_\rho^*(t) \mathrm{d}X_t.$$

exists then Y is infinitely divisible with Lévy measure $\Upsilon_\rho(M)$.

For the integral to exist, it is necessary that $\Upsilon_\rho(M) \in \mathfrak{M}^0$.

When $A = 0$, $b = 0$ and M is symmetric this is also sufficient.

Domains and Ranges of Upsilon Transforms

For an upsilon transform Υ_ρ we define its domain to be

$$\mathcal{D}(\Upsilon_\rho) = \{M \in \mathfrak{M}_{sf0} : \Upsilon_\rho M \in \mathfrak{M}^0\}$$

and its range to be

$$\mathcal{R}(\Upsilon_\rho) = \{\Upsilon_\rho M : M \in \mathcal{D}(\Upsilon_\rho)\}.$$

Domains and Ranges of Upsilon Transforms

For an upsilon transform Υ_ρ we define its domain to be

$$\mathcal{D}(\Upsilon_\rho) = \{M \in \mathfrak{M}_{sf0} : \Upsilon_\rho M \in \mathfrak{M}^0\}$$

and its range to be

$$\mathcal{R}(\Upsilon_\rho) = \{\Upsilon_\rho M : M \in \mathcal{D}(\Upsilon_\rho)\}.$$

Necessarily $\mathcal{D}(\Upsilon_\rho) \subset \mathfrak{M}^0$.

General Result on Domains and Ranges

Lemma

Assume that $\rho(\mathrm{d}s) = g(s)1_{s>0}\mathrm{d}s$ *and that there exist* $\delta \in (0,1)$, $\alpha \in \mathbb{R}$, *and* $0 < a < b < \infty$ *such that* $a < s^{\alpha+1}g(s) < b$ *for all* $s \in (0,\delta)$. *When* $\alpha < 2$ *assume also that* $\int_0^\infty s^2 g(s)\mathrm{d}s < \infty$. *In this case*

$$\mathfrak{D}(\Upsilon_\rho) = \begin{cases} \mathfrak{M}^0 & \text{if } \alpha < 0 \\ \mathfrak{M}^{\log} & \text{if } \alpha = 0 \\ \mathfrak{M}^\alpha & \text{if } \alpha \in (0,2) \\ \{0\} & \text{if } \alpha \geq 2 \end{cases}.$$

Further, when $\alpha \in (0,2)$ *we have* $\mathfrak{R}(\Upsilon_\rho) \subset \mathfrak{M}^\beta$ *for every* $\beta \in [0,\alpha)$.

First Transform

Consider the dilation measure:

1. For $\alpha \in \mathbb{R}$ and $p > 0$ let

$$\psi_{\alpha,p}(\mathrm{d}s) = s^{-\alpha-1} e^{-s^p} 1_{s>0} \mathrm{d}s.$$

We write

$$\Psi_{\alpha,p} = \Upsilon_{\psi_{\alpha,p}}.$$

First Transform

Consider the dilation measure:

1. For $\alpha \in \mathbb{R}$ and $p > 0$ let

$$\psi_{\alpha,p}(\mathrm{d}s) = s^{-\alpha-1} e^{-s^p} 1_{s>0} \mathrm{d}s.$$

We write

$$\Psi_{\alpha,p} = \Upsilon_{\psi_{\alpha,p}}.$$

For $p = 1$ introduced in Sato (2006). General case introduced in Maejima and Nakahara (2009).

Domain of $\Psi_{\alpha,p}$

For $\alpha \in \mathbb{R}$ and $p > 0$ we have

$$\mathfrak{D}(\Psi_{\alpha,p}) = \begin{cases} \mathfrak{M}^0 & \text{if } \alpha < 0 \\ \mathfrak{M}^{\log} & \text{if } \alpha = 0 \\ \mathfrak{M}^\alpha & \text{if } \alpha \in (0,2) \\ \{0\} & \text{if } \alpha \geq 2 \end{cases}$$

On this domain $\Psi_{\alpha,p}$ is a one-to-one transform.

Range of $\Psi_{\alpha,p}$

If $M \in \mathfrak{D}(\Psi_{\alpha,p})$ then for $B \in \mathfrak{B}(\mathbb{R}^d)$

$$[\Psi_{\alpha,p}M](B) = \int_{\mathbb{R}^d}\int_0^\infty 1_B(xt)t^{-1-\alpha}e^{-t^p}\mathrm{d}tM(\mathrm{d}x).$$

Range of $\Psi_{\alpha,p}$

If $M \in \mathfrak{D}(\Psi_{\alpha,p})$ then for $B \in \mathfrak{B}(\mathbb{R}^d)$

$$[\Psi_{\alpha,p} M](B) = \int_{\mathbb{R}^d} \int_0^\infty 1_B(xt) t^{-1-\alpha} e^{-t^p} \mathrm{d}t M(\mathrm{d}x).$$

This is the Lévy measure of a p-tempered α-stable distribution.

Range of $\Psi_{\alpha,p}$

If $M \in \mathfrak{D}(\Psi_{\alpha,p})$ then for $B \in \mathfrak{B}(\mathbb{R}^d)$

$$[\Psi_{\alpha,p}M](B) = \int_{\mathbb{R}^d} \int_0^\infty 1_B(xt) t^{-1-\alpha} e^{-t^p} \mathrm{d}t M(\mathrm{d}x).$$

This is the Lévy measure of a p-tempered α-stable distribution.

In this context M is called the Rosiński measure.

Range of $\Psi_{\alpha,p}$

If $M \in \mathfrak{D}(\Psi_{\alpha,p})$ then for $B \in \mathfrak{B}(\mathbb{R}^d)$

$$[\Psi_{\alpha,p}M](B) = \int_{\mathbb{R}^d}\int_0^\infty 1_B(xt)t^{-1-\alpha}e^{-t^p}\mathrm{d}tM(\mathrm{d}x).$$

This is the Lévy measure of a p-tempered α-stable distribution.

In this context M is called the Rosiński measure.

For $p = 1$ these were introduced in Rosiński (2007), the general case was introduced in Grabchak (2012), see also Grabchak (2016).

Second Transform

Consider the dilation measure:

2. For $-\infty < \beta < \alpha < \infty$ and $p > 0$ let

$$\tau_{\beta\to\alpha,p}(\mathrm{d}s) = \frac{1}{K_{\alpha,\beta,p}} s^{-\alpha-1}(1-s^p)^{(\alpha-\beta)/p-1} 1_{0<s<1}\mathrm{d}s,$$

where

$$K_{\alpha,\beta,p} = \int_0^\infty u^{\alpha-\beta-1} e^{-u^p}\mathrm{d}u = p^{-1}\Gamma\left(\frac{\alpha-\beta}{p}\right).$$

We write

$$\mathfrak{T}_{\beta\to\alpha,p} = \Upsilon_{\tau_{\beta\to\alpha,p}}.$$

Second Transform

Consider the dilation measure:

2. For $-\infty < \beta < \alpha < \infty$ and $p > 0$ let

$$\tau_{\beta\to\alpha,p}(\mathrm{d}s) = \frac{1}{K_{\alpha,\beta,p}} s^{-\alpha-1}(1-s^p)^{(\alpha-\beta)/p-1} 1_{0<s<1}\mathrm{d}s,$$

where

$$K_{\alpha,\beta,p} = \int_0^\infty u^{\alpha-\beta-1} e^{-u^p}\mathrm{d}u = p^{-1}\Gamma\left(\frac{\alpha-\beta}{p}\right).$$

We write

$$\mathfrak{T}_{\beta\to\alpha,p} = \Upsilon_{\tau_{\beta\to\alpha,p}}.$$

For $p = 1$ introduced in Sato (2006). General case introduced in Maejima et al. (2013).

Domain of $\mathfrak{T}_{\beta\to\alpha,p}$

For $-\infty < \beta < \alpha < \infty$ and $p > 0$ we have

$$\mathfrak{D}(\mathfrak{T}_{\beta\to\alpha,p}) = \mathfrak{D}(\Psi_{\alpha,p}) = \begin{cases} \mathfrak{M}^0 & \text{if } \alpha < 0 \\ \mathfrak{M}^{\log} & \text{if } \alpha = 0 \\ \mathfrak{M}^\alpha & \text{if } \alpha \in (0,2) \\ \{0\} & \text{if } \alpha \geq 2 \end{cases}.$$

On this domain $\mathfrak{T}_{\beta\to\alpha,p}$ is a one-to-one transform.

Third Transform

Consider the dilation measure:

3. For $0 < q < p < \infty$ and $\alpha \in \mathbb{R}$ let

$$\pi_{\alpha,p\to q}(\mathrm{d}s) = p f_{q/p}(s^{-p}) s^{-\alpha-p-1} 1_{s>0} \mathrm{d}s,$$

where, for $r \in (0,1)$, f_r is the density of a fully right skewed r-stable distribution with Laplace transform

$$\int_0^\infty e^{-tx} f_r(x) \mathrm{d}x = e^{-t^r}.$$

We write

$$\mathfrak{P}_{\alpha,p\to q} = \Upsilon_{\pi_{\alpha,p\to q}}.$$

Domain of $\mathfrak{P}_{\alpha,p\to q}$

For $\alpha \in \mathbb{R}$ and $0 < q < p < \infty$ we have

$$\mathfrak{D}(\mathfrak{P}_{\alpha,p\to q}) = \begin{cases} \mathfrak{M}^0 & \text{if } \alpha - q < 0 \\ \mathfrak{M}^{\log} & \text{if } \alpha - q = 0 \\ \mathfrak{M}^{\alpha-q} & \text{if } \alpha - q \in (0,2) \\ \{0\} & \text{if } \alpha - q \geq 2 \end{cases}.$$

On this domain, when $\alpha < 0$, $\mathfrak{P}_{\alpha,p\to q}$ is a one-to-one transform.

The case $\alpha \geq 0$ remains an open question.

However, it can be shown that $\mathcal{R}(\Psi_{\alpha,p}) \subsetneq \mathfrak{D}(\mathfrak{P}_{\alpha,p\to q})$ and as a mapping from $\mathcal{R}(\Psi_{\alpha,p})$, $\mathfrak{P}_{\alpha,p\to q}$ is a one-to-one transform.

Compositions of Transforms

For two upsilon transforms $\Upsilon_{\rho_1}, \Upsilon_{\rho_2}$ define the composition $\Upsilon_{\rho_2}\Upsilon_{\rho_1}$ on the domain

$$\mathfrak{D}(\Upsilon_{\rho_2}\Upsilon_{\rho_1}) = \{M \in \mathfrak{D}(\Upsilon_{\rho_1}) : \Upsilon_{\rho_1} M \in \mathfrak{D}(\Upsilon_{\rho_2})\}.$$

Then $\mathfrak{D}(\Upsilon_{\rho_2}\Upsilon_{\rho_1}) = \mathfrak{D}(\Upsilon_{\rho_1}\Upsilon_{\rho_2})$ and

$$\Upsilon_{\rho_2}\Upsilon_{\rho_1} = \Upsilon_{\rho_1}\Upsilon_{\rho_2}.$$

Thus compositions of upsilon transforms commute.

Compositions of Transforms

Theorem

1. If $-\infty < \beta < \alpha < 2$ *and* $p > 0$ *then*

$$\Psi_{\alpha,p} = \mathfrak{T}_{\beta\to\alpha,p}\Psi_{\beta,p}.$$

Compositions of Transforms

Theorem

1. *If* $-\infty < \beta < \alpha < 2$ *and* $p > 0$ *then*

$$\Psi_{\alpha,p} = \mathfrak{T}_{\beta\to\alpha,p}\Psi_{\beta,p}.$$

2. *If* $-\infty < \gamma < \beta < \alpha < 2$ *and* $p > 0$ *then*

$$\mathfrak{T}_{\gamma\to\alpha,p} = \mathfrak{T}_{\beta\to\alpha,p}\mathfrak{T}_{\gamma\to\beta,p}.$$

Compositions of Transforms

Theorem

1. *If* $-\infty < \beta < \alpha < 2$ *and* $p > 0$ *then*

$$\Psi_{\alpha,p} = \mathfrak{T}_{\beta\to\alpha,p}\Psi_{\beta,p}.$$

2. *If* $-\infty < \gamma < \beta < \alpha < 2$ *and* $p > 0$ *then*

$$\mathfrak{T}_{\gamma\to\alpha,p} = \mathfrak{T}_{\beta\to\alpha,p}\mathfrak{T}_{\gamma\to\beta,p}.$$

3. *If* $\alpha < 2$ *and* $0 < q < p < \infty$ *then*

$$\Psi_{\alpha,q} = \mathfrak{P}_{\alpha,p\to q}\Psi_{\alpha,p}.$$

Compositions of Transforms

Theorem

1. *If* $-\infty < \beta < \alpha < 2$ *and* $p > 0$ *then*

$$\Psi_{\alpha,p} = \mathfrak{T}_{\beta\to\alpha,p}\Psi_{\beta,p}.$$

2. *If* $-\infty < \gamma < \beta < \alpha < 2$ *and* $p > 0$ *then*

$$\mathfrak{T}_{\gamma\to\alpha,p} = \mathfrak{T}_{\beta\to\alpha,p}\mathfrak{T}_{\gamma\to\beta,p}.$$

3. *If* $\alpha < 2$ *and* $0 < q < p < \infty$ *then*

$$\Psi_{\alpha,q} = \mathfrak{P}_{\alpha,p\to q}\Psi_{\alpha,p}.$$

4. *If* $0 < r < q < p < \infty$ *and* $-\infty < \alpha < 2 + r$ *then*

$$\mathfrak{P}_{\alpha,p\to r} = \mathfrak{P}_{\alpha,q\to r}\mathfrak{P}_{\alpha,p\to q}.$$

Main Theorem for p-Tempered β-Stable Lévy Measures

If $-\infty < \beta < \alpha < 2$ and $0 < q < p < \infty$, then for $M \in \mathfrak{M}^\alpha$ and $B \in \mathfrak{B}(\mathbb{R}^d)$

$$\mathfrak{T}_{\beta\to\alpha,p}\left(\int_{\mathbb{R}^d}\int_0^\infty 1_B(xt)t^{-1-\beta}e^{-t^p}\mathrm{d}tM(\mathrm{d}x)\right)$$

Main Theorem for p-Tempered β-Stable Lévy Measures

If $-\infty < \beta < \alpha < 2$ and $0 < q < p < \infty$, then for $M \in \mathfrak{M}^\alpha$ and $B \in \mathfrak{B}(\mathbb{R}^d)$

$$\mathfrak{T}_{\beta\to\alpha,p}\left(\int_{\mathbb{R}^d}\int_0^\infty 1_B(xt)t^{-1-\beta}e^{-t^p}\mathrm{d}tM(\mathrm{d}x)\right) = \int_{\mathbb{R}^d}\int_0^\infty 1_B(xt)t^{-1-\alpha}e^{-t^p}\mathrm{d}tM(\mathrm{d}x)$$

Main Theorem for p-Tempered β-Stable Lévy Measures

If $-\infty < \beta < \alpha < 2$ and $0 < q < p < \infty$, then for $M \in \mathfrak{M}^\alpha$ and $B \in \mathfrak{B}(\mathbb{R}^d)$

$$\mathfrak{T}_{\beta\to\alpha,p}\left(\int_{\mathbb{R}^d}\int_0^\infty 1_B(xt)t^{-1-\beta}e^{-t^p}\mathrm{d}tM(\mathrm{d}x)\right) = \int_{\mathbb{R}^d}\int_0^\infty 1_B(xt)t^{-1-\alpha}e^{-t^p}\mathrm{d}tM(\mathrm{d}x)$$

and

$$\mathfrak{P}_{\beta,p\to q}\left(\int_{\mathbb{R}^d}\int_0^\infty 1_B(xt)t^{-1-\beta}e^{-t^p}\mathrm{d}tM(\mathrm{d}x)\right) = \int_{\mathbb{R}^d}\int_0^\infty 1_B(xt)t^{-1-\beta}e^{-t^q}\mathrm{d}tM(\mathrm{d}x)$$

Bibliography

O. Barndorff-Nielsen, J. Rosiński, and S. Thorbjørnsen (2008). "General Υ transforms." ALEA 4:131-165.

M. Grabchak (2015). "Three Upsilon Transforms Related to Tempered Stable Distributions." *Electronic Communications in Probability*, 20(82):1–10.

M. Grabchak (2016). *Tempered Stable Distributions–Stochastic Models for Multiscale Processes.* Springer, Cham, Switzerland.

J. Rosiński (2007). "Tempering stable processes." *Stochastic Processes and their Applications*, 117(6):677-707.

Domains and Ranges

If $-\infty < \gamma < \beta < \alpha < \infty$ and $0 < r < q < p < \infty$ then

$$\begin{aligned}
\mathfrak{R}(\mathfrak{T}_{\beta\to\alpha,p}) &\subset \mathfrak{D}(\Psi_{\beta,p}),\\
\mathfrak{R}(\mathfrak{T}_{\beta\to\alpha,p}) &\subset \mathfrak{D}(\mathfrak{T}_{\gamma\to\beta,p}),\\
\mathfrak{R}(\Psi_{\alpha,p}) &\subset \mathfrak{D}(\mathfrak{P}_{\alpha,p\to q}),\\
\mathfrak{R}(\mathfrak{P}_{\alpha,q\to r}) &\subset \mathfrak{D}(\mathfrak{P}_{\alpha,p\to q}),\\
\mathfrak{R}(\mathfrak{T}_{\beta\to\alpha,p}) &\subset \mathfrak{D}(\mathfrak{P}_{\alpha,p\to q}).
\end{aligned}$$