On Copulæ and their Applications

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1 Introduction

Let $\mathbf{X} := (X_1, \ldots, X_d)$ be a random vector in $\mathbb{R}^d$, defined on the probability space $(\Omega, \mathcal{A}, \mathbb{P})$. We denote by $\mathbb{F}(\mathbf{x}) := \mathbb{P}(\mathbf{X} \leq \mathbf{x}) = \mathbb{P}(X_1 \leq x_1, \ldots, X_d \leq x_d)$, for $\mathbf{x} = (x_1, \ldots, x_d) \in \mathbb{R}^d$, the joint distribution function of $\mathbf{X}$. Here, for $\mathbf{x} = (x_1, \ldots, x_d) \in \mathbb{R}^d$ and $\mathbf{y} = (y_1, \ldots, y_d) \in \mathbb{R}^d$, we write $\mathbf{x} \leq \mathbf{y}$ to denote the event that $x_j \leq y_j$ for $j = 1, \ldots, d$. For $x \in \mathbb{R}$, denote by $\mathbb{F}^{(j)}(x) := \mathbb{P}(X_j \leq x)$, $j = 1, \ldots, d$ the marginal distribution functions of $\mathbf{X} := (X_1, \ldots, X_d)$. Introduce the corresponding marginal quantile functions, defined by $\mathbb{Q}^{(j)}(t) := \inf\{x : \mathbb{F}^{(j)}(x) \geq t\}$, for $0 < t < 1$ and $j = 1, \ldots, d$. By a copula function is meant the joint distribution function $\mathbb{C}(\mathbf{u}) := \mathbb{P}(\mathbf{U} \leq \mathbf{u}) = \mathbb{P}(U_1 \leq u_1, \ldots, U_d \leq u_d)$, for $\mathbf{u} = (u_1, \ldots, u_d) \in \mathbb{R}^d$, of a random vector $\mathbf{U} = (U_1, \ldots, U_d) \in [0, 1]^d$, with uniform marginals, namely, such that $\mathbb{P}(U_j \leq u) = u$ for $0 \leq u \leq 1$ and $j = 1, \ldots, d$. The fundamental theorem of *copula theory* may now be stated as follows.

Theorem 1.1. *On a suitably enlarged version of the probability space* $(\Omega, \mathcal{A}, \mathbb{P})$*, it is possible to define jointly* $\mathbf{X} := (X_1, \ldots, X_d) \in \mathbb{R}^d$ *and a random vector* $\mathbf{U} = (U_1, \ldots, U_d) \in [0, 1]^d$*, with uniform marginals, in such a way that, almost surely,*

$$\mathbf{X} = (X_1, \ldots, X_d) = \mathbb{Q}(\mathbf{U}) := (\mathbb{Q}^{(1)}(U_1), \ldots, \mathbb{Q}^{(d)}(U_d)). \tag{1.1}$$

Moreover, letting $\mathbb{C}(\mathbf{u}) := \mathbb{P}(\mathbf{U} \leq \mathbf{u}) = \mathbb{P}(U_1 \leq u_1, \ldots, U_d \leq u_d)$ *denote the copula function of* $\mathbf{X}$*, defined as the distribution function of* $\mathbf{U}$*, we have the identity, for* $\mathbf{x} = (x_1, \ldots, x_d) \in \mathbb{R}^d$*,*

$$\mathbb{F}(\mathbf{x}) = \mathbb{F}(x_1, \ldots, x_d) = \mathbb{C}(\mathbb{F}^{(1)}(x_1), \ldots, \mathbb{F}^{(d)}(x_d)). \tag{1.2}$$

The relation (1.1) is known under the name of *Sklar's theorem* (refer to [1]). The copula function $\mathbb{C}(\cdot)$ of $\mathbf{X}$ is uniquely defined iff the marginal distribution functions $\mathbb{F}^{(1)}, \ldots, \mathbb{F}^{(d)}$ of $\mathbf{X}$ are continuous.

Via Theorem 1.1, copula functions provide a universal tool to model the dependence relationships between the coordinates of arbitrary random vectors. The aim of the present course is to highlight some of the main aspects and applications of the theory of copulæ, where a number of open problems remain unsolved. Because of their flexibility, in the recent years, copulae have been extensively used in economics, insurance and finance.

2 Tentative Contents of the Course

Chapter 1: General theory of copulæ - Existence - Bounds - Examples
Chapter 2: Statistical Inference - The Copula Process - Limit Theorems
Chapter 3: Tests of Independence - Measures of Dependence
Chapter 4: A Collection of Copulæ of Practical Interest
Chapter 5: Extreme-Value Copulæ - Statistical Inference
Chapter 6: Archimedean Copulæ - Statistical Inference
Chapter 7: Farlie-Gumbel-Morgenstern Models
Chapter 8: Local Models - Truncation - Tail Dependence
Chapter 9: Modelling Dependence by Copulæ - Applications
Chapter 10: Alternative to Copulæ - The Rosenblatt Transformation

References

[1] Sklar, M. (1959). Fonctions de répartition à N dimensions et leurs marges. *Publ. Inst. Statist. Paris*, **8**, 229-231.